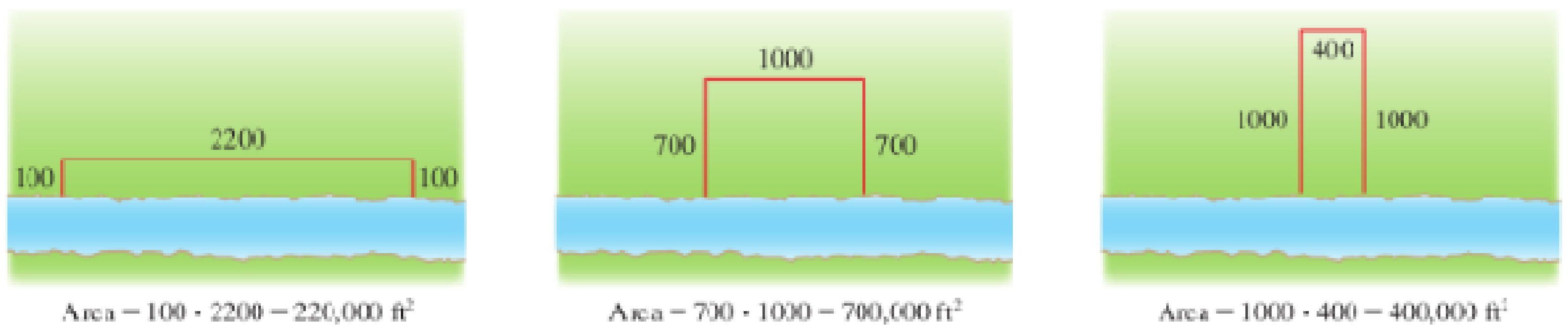
4.7. Optimization Problems

1 **Understand the Problem** The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?

2 **Draw a Diagram** In most problems it i

given and required quantities on the d

Figure 1



3 **Introduce Notation** Assign a symbol to the quantity that is to be maximized or minimized (let's call it Q for now). Also select symbols $(a, b, c, \ldots, x, y)$ for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols—for example, A for area, h for height, t for time.

4 Express Q in terms of some of the other symbols from Step 3.

5 If Q has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q. Thus Q will be expressed as a function of *one* variable x, say, $Q = f(x)$. Write the domain of this function in the given context.

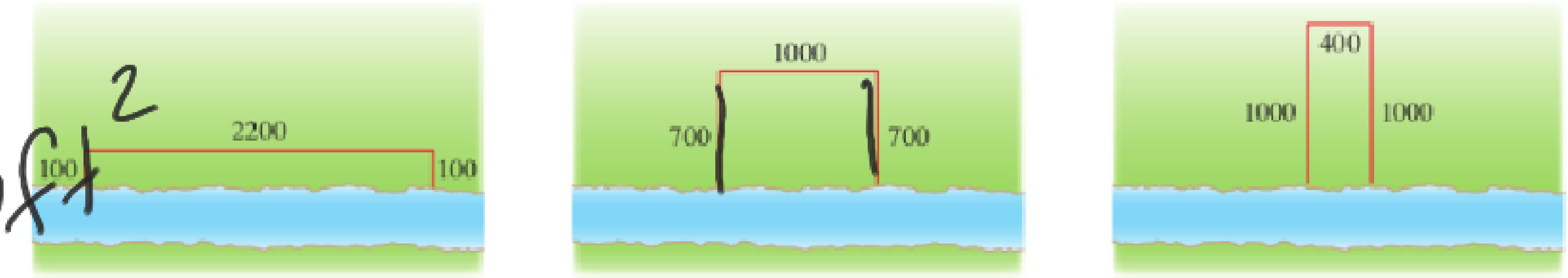
6 Use the methods of Sections 4.1 and 4.3 to find the *absolute* maximum or minimum value of f. In particular, if the domain of f is a closed interval, then the Closed Interval Method in Section 4.1 can be used.

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

$\boxed{\begin{aligned} W &= 600 \\ L &= 1200 \end{aligned}}$

$A = 720{,}000\ ft^2$

Figure 1



Area = 100 · 2200 = 220,000 ft²   Area = 700 · 1000 = 700,000 ft²   Area = 1000 · 400 = 400,000 ft²

$\boxed{0 \le W \le 1200}$

$2w + L = 2400$

$L = 2400 - 2w$

$A = LW$

$A = w(2400 - 2w)$

$A = 2400w - 2w^2 \qquad 0 \le w \le 1200$

$A' = 2400 - 4w \qquad 0 < 2400 - 4w$

$0 = 2400 - 4w \qquad 4w < 2400 \quad A' > 0$ positive

$4w = 2400 \qquad w < 600$

$\boxed{w = 600}$ max $\qquad w > 600 \quad A' < 0$ negative

A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

$$A = 2\pi r^2 + 2\pi r h$$

Top + Bottom

Side

$$V = \pi r^2 h$$

$$1000\ cm^3 = \pi r^2 h$$

$$\frac{1000}{\pi r^2} = h$$

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$A = 2\pi r^2 + \frac{2000}{r}$$

$$A = 2\pi r^2 + 2000r^{-1}$$

$$\boxed{r > 0}$$

$$A' = 4\pi r - \frac{2000}{r^2}$$

$$0 = 4\pi r - \frac{2000}{r^2} = \frac{4\pi r^3 - 2000}{r^2} = 0$$

$$4\pi r^3 - 2000 = 0$$

$$4\pi r^3 = 2000$$

$$r^3 = 500/\pi$$

$$r = \sqrt[3]{\frac{500}{\pi}}$$

$$\frac{4\pi r^3 - 2000}{r^2} > 0$$

$$4\pi r^3 - 2000 > 0$$

$$4\pi r^3 > 2000$$

$$r^3 > 500/\pi$$

$$r > \sqrt[3]{500/\pi}$$

Positive

minimum

$$\boxed{r = \sqrt[3]{500/\pi}}$$

$$\frac{4\pi r^3 - 2000}{r^2} < 0$$

$$4\pi r^3 - 2000 < 0$$

$$4\pi r^3 < 2000$$

$$r^3 < 500/\pi$$

$$r < \sqrt[3]{500/\pi}$$

Negative

$$h = \frac{1000}{\pi r^2}$$

$$\boxed{h = 2r}$$

$$h = \frac{1000}{\pi\left(\frac{500}{\pi}\right)^{2/3}} = \boxed{2\sqrt[3]{\frac{500}{\pi}}}$$

$$h = \frac{\sqrt[3]{1000^3}}{\sqrt[3]{\pi^3}\sqrt[3]{\left(\frac{500^2}{\pi^2}\right)}} = \sqrt[3]{\frac{1000^3}{\pi^3\left(\frac{(500)^2}{\pi^2}\right)}} = 2\sqrt[3]{\frac{500}{\pi}}$$

Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$(1, 4)$

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

$y^2 = 2x$

$$d = \sqrt{\left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2}$$

$\frac{y^2}{2} = x$

$$d = \sqrt{\left(\frac{y^2}{2}-1\right)^2 + (y-4)^2}$$

$$d^2 = \boxed{\left(\frac{y^2}{2}-1\right)^2 + (y-4)^2}$$

$$f'(y) = (y)\left(2\left(\frac{y^2}{2}-1\right)\right) + (1)\left(2(y-4)\right)$$

$$= 2y\left(\frac{y^2}{2}-1\right) + 2(y-4)$$

$$= y^3 - 2y + 2y - 8 = y^3 - 8$$

$f'(y) = y^3 - 8$

$y^3 - 8 > 0$ postive

$y > 2$

$y^3 - 8 < 0$ Negative

$y < 2$

$y^3 - 8 = 0$

$y^3 = 8$

$y = 2$

Minimum

$\boxed{(2, 2)}$

$y^2 = 2x$

$4 = 2x$

$2 = x$

First Derivative Test for Absolute Extreme Values

Suppose that c is a critical number of a continuous function f defined on an interval.

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

A store has been selling 200 TV monitors a week at \$350 each. A market survey indicates that for each \$10 rebate offered to buyers, the number of monitors sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize revenue?

$$\frac{10}{20} = \frac{1}{2} \qquad P(x) = 350 - \frac{1}{2}(x - 200)$$

Price $\rightarrow$ $P(x) = 450 - \frac{1}{2}x$

Revenue $\rightarrow$ $R(x) = x\left(450 - \frac{1}{2}x\right)$

$$= 450x - \frac{1}{2}x^2$$

$$\boxed{R(x) = 450x - \frac{1}{2}x^2}$$

$$R'(x) = 450 - x$$

$$450 - x = 0$$

$$-x = -450$$

$$\boxed{x = 450}$$

Sell 450 monitors

max

$$P(450) = 480 - \frac{1}{2}(450$$

$$P(450) = \$225$$

Rebate $350 - 225$

$= \$125$ off