Chapter 4 Review Test
Name:

MATK254 Calculus 1
Date: 11/15/2022

For problems number 1 – 6 find the critical numbers and identify if they are minimum or maximum then identify which interval the function is concave up or concave down.

1) $y = x^3 + 3x^2$

$y' = 3x^2 + 6x$

Critical #'s

$0 = 3x(x+2)$

$3x = 0 \qquad x + 2 = 0$

$x = 0 \qquad x = -2$

$y'' = 6x + 6$

Crtical #'s

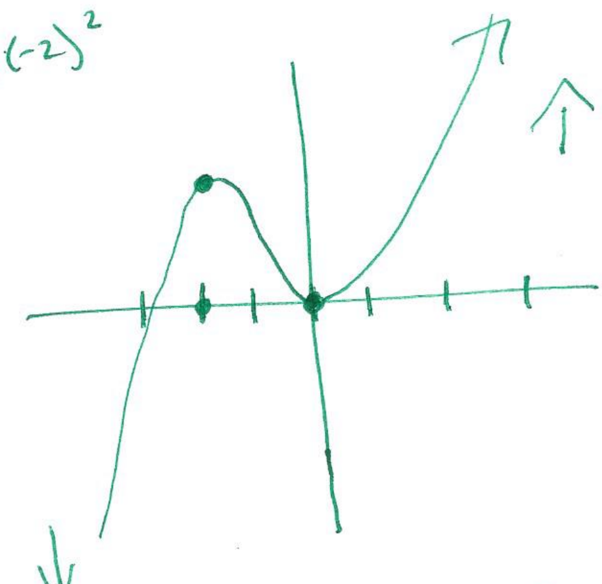
$0 = 6(x+1)$

$0 = x + 1$

$x = -1$

$y = (-2)^3 + 3(-2)^2$
$-8 + 12$
4

$y = (0)^3 + 3(0)$
$= 0$



Local Max $(-2, 4)$

Local Min $(0, 0)$

Concave down
$(-\infty, -1)$

Concave up
$(-1, \infty)$

2) $y = \frac{x-x^2}{2-3x+x^2} = \frac{x(1-x)}{(x-2)(x-1)} = \frac{-x(x-1)}{(x-2)(x-1)} = \frac{-x}{x-2}$

$y = \frac{-x}{x-2}$ $\quad f(x) = -x \quad f'(x) = -1$

$g(x) = x-2 \quad g'(x) = 1$

$y' = \frac{-1(x-2) - (1)(-x)}{(x-2)^2} = \frac{-x+2+x}{(x-2)^2} = \frac{2}{(x-2)^2}$

Critical #'s $\quad x-2=0$

$x = 2$

Since $y' > 0$ on all x's there is no min or max

$f(x) = 2 \quad f'(x) = 0$

$g(x) = (x-2)^2 \quad g'(x) = 2(x-2)$

$y'' = \frac{0(x-2)^2 - 2(2(x-2))}{(x-4)^4} = \frac{-4(x-2)}{(x-2)^4} = \frac{-4}{(x-2)^3}$

Critical #'s $\quad x-2=0$

$x=2$

$y'' > 0$ when $x < 2$ so y is concave up $(-\infty, 2)$

when $x > 2$ so y is concave down $(2, \infty)$

$y'' < 0$

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3) $y = (x-4)\sqrt[3]{x} = x^{4/3} - 4x^{1/3}$

$y' = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4x^{1/3}}{3} - \frac{4}{3x^{2/3}} = \frac{4x-4}{3x^{2/3}}$

Critical #'s

$4x-4=0$
$4x=4$
$x=1$

$3x^{2/3}=0$
$x^{2/3}=0$
$x=0$

$y'<0 \quad x<1$
$y'>0 \quad x>1$ so $(1,-3)$ min

$y=(1-4)\sqrt[3]{1} = -3$

$y'' \rightarrow \quad f(x) = 4x-4 \qquad f'(x) = 4$
$g(x) = 3x^{2/3} \qquad g'(x) = 2x^{-1/3}$

$y'' = \frac{4(3x^{2/3}) - (2x^{-1/3}(4x-4))}{9x^{4/3}}$

$= \frac{12x^{2/3} - 8x^{2/3} + 8x^{-1/3}}{9x^{4/3}} = 4x^{2/3} + \frac{8}{x^{1/3}} = \frac{\frac{4x+8}{x^{1/3}}}{9x^{4/3}}$

$y'' = \frac{4x+8}{9x^{5/3}}$

Critical #'s
$4x+8=0$
$x=-2$

$9x^{5/3}=0$
$x=0$

$y''>0 \quad x<-2 \quad (-\infty,-2)$ concave up
$y''<0 \quad -2<x<0 \quad (-2,0)$ concave down
$y''>0 \quad x>0 \quad (0,\infty$ concave up

4) $y = \sin^3 x$

$u = \sin x \quad u' = \cos x$

$y' = 3\sin^2 x \cos x$

Critical #'s

$\sin^2 x = 0 \qquad \cos x = 0$

$x = 0, \pi \qquad x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\sin^2 x > 0 \qquad \cos x > 0$

always $\qquad 0 < x < \frac{\pi}{2} \qquad \frac{3\pi}{2} < x < 2\pi$

$\cos x < 0$

$\frac{\pi}{2} < x < \frac{3\pi}{2}$

So $y' > 0 \quad 0 < x \frac{\pi}{2}$ and $\frac{3\pi}{2} < x < 2\pi$

$y' < 0 \quad \frac{\pi}{2} < x < \frac{3\pi}{2}$

max at $x = \frac{\pi}{2} \quad \left(\frac{\pi}{2}, 1\right)$

min at $x = \frac{3\pi}{2} \quad \left(\frac{3\pi}{2}, -1\right)$

No Concavity as it changes multiple times

5) $y = \frac{1}{1+e^{-x}} = (1+e^{-x})^{-1}$

$u = 1 + e^{-x} \qquad u' = -e^{-x}$

$y' = (-e^{-x})(-1(1+e^{-x})^{-2})$

$y' = \frac{e^{-x}}{(1+e^{-x})^2} \qquad e^{-x} \neq 0$

$1 + e^{-x} \neq 0$

So no critical #'s

$y'' \rightarrow f(x) = e^{-x} \qquad f'(x) = -e^{x}$

$g(x) = (1+e^{-x})^2 \quad g'(x) = -2e^{-x}(1+e^{-x})$

$y'' = \frac{-e^{-x}(1+e^{-x})^2 - (-2e^{-x}(1+e^{-x}))e^{-x}}{(1+e^{-x})^4}$

$y'' = \frac{e^{-2x}(e^{x}+1)}{(1+e^{-x})^3}$

Critical #'s

$e^{-2x} \neq 0 \qquad -e^{x}+1 = 0$

$(1+e^{-x})^3 \neq 0 \qquad x = 0$

$y'' < 0 \quad x < 0 \quad$ concave down $(-\infty, 0)$

$y'' > 0 \quad x > 0 \quad$ concave up $(0, \infty)$

6) $y = x(\ln x)^2$

y' $\quad f(x) = x \quad f'(x) = 1$

$g(x) = (\ln x)^2 \quad g'(x) = \frac{2\ln(x)}{x}$

$y' = (\ln x)^2 + \frac{2x\ln x}{x} = (\ln x)^2 + 2\ln x$

Critical #'s

$(\ln x)^2 + 2\ln x = 0$

$u = \ln x$

$u^2 + 2u = 0$

$u(u+2) = 0$

$u = 0 \quad u = -2$

$\ln x = 0 \quad \ln x = -2$

$x = 1 \quad x = e^{-2} = \frac{1}{e^2}$

$y' > 0 \quad x < \frac{1}{e^2}$

$y' < 0 \quad \frac{1}{e^2} < x < 1$

$y' > 0 \quad x > 1$

max at $x = \frac{1}{e^2}$

min at $x = 1$

$y'' = \frac{2\ln(x)}{x} + \frac{2}{x} = \frac{2(\ln x + 1)}{x}$

Critical #s

$2\ln x + 2 = 0$

$\ln x = -1$

$x = e^{-1}$

$x = \frac{1}{e}$

$x = 0$ not in domain

$y'' < 0 \quad x < 1/e$ concave down

$y'' > 0 \quad x > 1/e$ concave up

Use l'Hospital's rule (if needed) to find the limit given in questions 7-8

7) $\lim_{x \to 4} \frac{x^2-2x-8}{x-4}$

$$\frac{x^2-2x-8}{x-4} = \frac{(x-4)(x+2)}{x-4}$$

$$\lim_{x \to 4} x+2 = \boxed{6}$$

8) $\lim_{x\to 0} \sin 5x \csc 3x$

$\csc 3x = \frac{1}{\sin 3x}$

$$\lim_{x\to 0} \frac{\sin 5x}{\sin 3x}$$

$f(x) = \sin 5x \quad f'(x) = 5\cos 5x$

$g(x) = \sin 3x \quad g'(x) = 3\cos 3x$

$$\lim_{x\to 0} \frac{5\cos 5x}{3\cos 3x} = \boxed{\frac{5}{3}}$$

9) A box with a square base and an open top must have a volume of 32,000cm^3. Find the dimensions of the box that minimize the amount of material used.

$V = LWH = 32000$ $\quad$ $L = W$ (square base)

$L^2 H = 32000$

$H = 32000/L^2$

$S.A. = L^2 + 4LH$

$= L^2 + 4L\left(\frac{32000}{L^2}\right)$

$y = L^2 + \frac{128000}{L}$

$y' = 2L - \frac{128000}{L^2} = \frac{2L^3 - 128000}{L^2}$

Critical #'s

$2L^3 - 128000 = 0$ $\quad$ $L^2 = 0$

$L^3 = 64000$ $\quad$ $L = 0$ (not in domain)

$$\boxed{\begin{array}{l} L = 40\text{ cm} \\ W = 40\text{ cm} \\ H = 32000/1600 = 20\text{cm} \end{array}}$$

10) Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point (1, 0).

$x_1\ y_1$
$(1, 0)$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$4x^2 + y^2 = 4$$
$$y^2 = 4 - 4x^2$$
$$y = \pm\sqrt{4 - 4x^2}$$

$$d = \sqrt{(x-1)^2 + (\pm\sqrt{4-4x^2} - 0)^2}$$

$$d^2 = (x-1)^2 + (4 - 4x^2)$$
$$= x^2 - 2x + 1 + 4 - 4x^2$$
$$= -3x^2 - 2x + 5$$

$f(x) = -3x^2 - 2x + 5 \qquad f'(x) = -6x - 2$

$$-6x - 2 = 0$$
$$-6x = 2$$
$$x = -\frac{1}{3}$$

$$y = \pm\sqrt{4 - 4\left(\tfrac{-1}{3}\right)^2}$$
$$= \pm\sqrt{4 - \tfrac{4}{9}}$$
$$= \pm\sqrt{\tfrac{32}{9}} = \pm\frac{4}{3}\sqrt{2}$$

$$\boxed{\left(-\frac{1}{3}, \frac{4\sqrt{2}}{3}\right) \quad \left(-\frac{1}{3}, -\frac{4\sqrt{2}}{3}\right)}$$