Chapter 7

Techniques of Integration

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7.2 Trigonometric Integrals

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Trigonometric Integrals (1 of 11)

In this section we use trigonometric identities to integrate certain combinations of trigonometric functions.

We start with powers of sine and cosine.

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Example 2

Find $\int \sin^5 x \cos^2 x\, dx$.

Solution:

We could convert $\cos^2 x$ to $1 - \sin^2 x$, but we would be left with an expression in terms of sin *x* with no extra cos *x* factor.

Instead, we separate a single sine factor and rewrite the remaining $\sin^4 x$ factor in terms of cos *x*:

$$\begin{aligned} \sin^5 x \cos^2 x &= (\sin^2 x)^2 \cos^2 x \sin x \\ &= (1 - \cos^2 x)^2 \cos^2 x \sin x \end{aligned}$$

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Example 2 – *Solution*

Substituting *u* = cos *x*, we have *du* = −sin *x dx* and so

$$\begin{aligned} \int \sin^5 x \cos^2 x\, dx &= \int (\sin^2 x)^2 \cos^2 x \sin x\, dx \\ &= \int (1 - \cos^2 x)^2 \cos^2 x \sin x\, dx \\ &= \int (1 - u^2)^2 u^2 (-du) = -\int (u^2 - 2u^4 + u^6)\, du \\ &= -\left(\frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7}\right) + C \\ &= -\tfrac{1}{3}\cos^3 x + \tfrac{2}{5}\cos^5 x - \tfrac{1}{7}\cos^7 x + C \end{aligned}$$

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Example 3

Evaluate $\int_0^\pi \sin^2 x\, dx$.

Solution:

If we write $\sin^2 x = 1 - \cos^2 x$, the integral is no simpler to evaluate. Using the half-angle formula for $\sin^2 x$, however, we have

$$\begin{aligned} \int_0^\pi \sin^2 x\, dx &= \tfrac{1}{2}\int_0^\pi (1 - \cos 2x)\, dx \\ &= [\tfrac{1}{2}(x - \tfrac{1}{2}\sin 2x)]_0^\pi \end{aligned}$$

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Example 3 – *Solution*

$$= \frac{1}{2}\left(\pi - \frac{1}{2}\sin 2\pi\right) - \frac{1}{2}\left(0 - \frac{1}{2}\sin 0\right)$$
$$= \frac{1}{2}\pi$$

Notice that we mentally made the substitution $u = 2x$ when integrating $\cos 2x$.

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Trigonometric Integrals (2 of 11)

To summarize, we list guidelines to follow when evaluating integrals of the form $\int \sin^m x \cos^n x \, dx$, where $m \geq 0$ and $n \geq 0$ are integers.

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Trigonometric Integrals (3 of 11)

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x (\cos^2 x)^k \cos x \, dx$$
$$= \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \sin x \, dx$$
$$= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx$$

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Trigonometric Integrals (4 of 11)

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \quad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2}\sin 2x$$

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Trigonometric Integrals (5 of 11)

We can use a similar strategy to evaluate integrals of the form $\int \tan^m x \sec^n x \, dx$.

Since $\left(\frac{d}{dx}\right) \tan x = \sec^2 x$, we can separate a $\sec^2 x$ factor and convert the remaining (even) power of secant to an expression involving tangent using the identity $\sec^2 x = 1 + \tan^2 x$.

Or, since $\left(\frac{d}{dx}\right) \sec x = \sec x \tan x$, we can separate a sec x tan x factor and convert the remaining (even) power of tangent to secant.

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Example 5

Evaluate $\int \tan^6 x \sec^4 x \, dx$.

Solution:

If we separate one $\sec^2 x$ factor, we can express the remaining $\sec^2 x$ factor in terms of tangent using the identity $\sec^2 x = 1 + \tan^2 x$.

We can then evaluate the integral by substituting $u = \tan x$ so that $du = \sec^2 x dx$:

$$\int \tan^6 x \sec^4 x \, dx = \int \tan^6 x \sec^2 x \sec^2 x \, dx$$

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Example 5 – *Solution*

$$= \int \tan^6 x(1+\tan^2 x)\sec^2 x\, dx$$
$$= \int u^6(1+u^2)du = \int (u^6+u^8)du$$
$$= \frac{u^7}{7} + \frac{u^9}{9} + C$$
$$= \tfrac{1}{7}\tan^7 x + \tfrac{1}{9}\tan^9 x + C$$

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Trigonometric Integrals (6 of 11)

The preceding examples demonstrate strategies for evaluating integrals of the form $\int \tan^m x \sec^n x\, dx$ for two cases, which we summarize here.

Strategy for Evaluating $\int \tan^m x \sec^n x\, dx$

(a) If the power of secant is even ($n = 2k$, $k \geq 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x\, dx = \int \tan^m x(\sec^2 x)^{k-1}\sec^2 x\, dx$$
$$= \int \tan^m x(1+\tan^2 x)^{k-1}\sec^2 x\, dx$$

Then substitute $u = \tan x$.

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Trigonometric Integrals (7 of 11)

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x\, dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x\, dx$$
$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x\, dx$$

Then substitute $u = \sec x$.

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Trigonometric Integrals (8 of 11)

For other cases, the guidelines are not as clear-cut. We may need to use identities, integration by parts, and occasionally a little ingenuity.

We will sometimes need to be able to integrate tan *x* by using the formula given below:

$$\int \tan x \; dx = \ln|\sec x| + C$$

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Trigonometric Integrals (9 of 11)

We will also need the indefinite integral of secant:

$$\textbf{1} \quad \int \sec x \; dx = \ln|\sec x + \tan x| + C$$

We could verify Formula 1 by differentiating the right side, or as follows. First we multiply numerator and denominator by sec *x* + tan *x*:

$$\int \sec x \; dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \; dx$$

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Trigonometric Integrals (10 of 11)

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \; dx$$

If we substitute *u* = sec *x* + tan *x*, then $du = (\sec x \tan x + \sec^2 x)\; dx$, so the integral becomes

$$\int \left(\frac{1}{u}\right) du = \ln|u| + C.$$

Thus we have

$$\int \sec x \; dx = \ln|\sec x + \tan x| + C$$

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Example 7

Find $\int \tan^3 x\, dx$.

Solution:

Here only tan *x* occurs, so we use $\tan^2 x = \sec^2 x - 1$ to rewrite a $\tan^2 x$ factor in terms of $\sec^2 x$:

$$\begin{aligned}\int \tan^3 x\, dx &= \int \tan x \tan^2 x\, dx \\ &= \int \tan x(\sec^2 x - 1)\, dx \\ &= \int \tan x \sec^2 x\, dx - \int \tan x\, dx\end{aligned}$$

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Example 7 – *Solution*

$$= \frac{\tan^2 x}{2} - \ln|\sec x| + C$$

In the first integral we mentally substituted $u = \tan x$ so that $du = \sec^2 x\, dx$.

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Trigonometric Integrals (11 of 11)

Finally, we can make use of another set of trigonometric identities:

2 To evaluate the integrals (a) $\int \sin mx \cos nx\, dx$, (b) $\int \sin mx \sin nx\, dx$, or (c) $\int \cos mx \cos nx\, dx$, use the corresponding identity:

(a) $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$

(b) $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$

(c) $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

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Example 9

Evaluate $\int \sin 4x \cos 5x \, dx$.

Solution:

This integral could be evaluated using integration by parts, but it's easier to use the identity in Equation 2(a) as follows:

$$\begin{aligned}\int \sin 4x \cos 5x \, dx &= \int \tfrac{1}{2}[\sin(-x) + \sin 9x] \, dx \\ &= \tfrac{1}{2}\int(-\sin x + \sin 9x) \, dx \\ &= \tfrac{1}{2}(\cos x - \tfrac{1}{9}\cos 9x) + C\end{aligned}$$

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7.3 Trigonometric Substitution

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Trigonometric Substitution (1 of 8)

In finding the area of a circle or an ellipse, an integral of the form $\int \sqrt{a^2 - x^2}\,dx$ arises, where $a > 0$.

If it were $\int x\sqrt{a^2 - x^2}\,dx$, the substitution $u = a^2 - x^2$ would be effective but, as it stands, $\int \sqrt{a^2 - x^2}\,dx$ is more difficult.

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Trigonometric Substitution (2 of 8)

If we change the variable from x to θ by the substitution $x = a \sin \theta$, then the identity $1 - \sin^2 \theta = \cos^2 \theta$ allows us to get rid of the root sign because

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2 \theta} \\ &= \sqrt{a^2 \left(1 - \sin^2 \theta\right)} \\ &= \sqrt{a^2 \cos^2 \theta} \\ &= a\left|\cos \theta\right|\end{aligned}$$

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Trigonometric Substitution (3 of 8)

Notice the difference between the substitution $u = a^2 - x^2$ (in which the new variable is a function of the old one) and the substitution $x = a \sin\theta$ (the old variable is a function of the new one).

In general, we can make a substitution of the form $x = g(t)$ by using the Substitution Rule in reverse.

To make our calculations simpler, we assume that g has an inverse function; that is, g is one-to-one.

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Trigonometric Substitution (4 of 8)

In this case, if we replace u by x and x by t in the Substitution Rule, we obtain

$$\int f(x)\,dx = \int f(g(t))\,g'(t)\,dt$$

This kind of substitution is called *inverse substitution*.

We can make the inverse substitution $x = a \sin \theta$ provided that it defines a one-to-one function.

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Trigonometric Substitution (5 of 8)

This can be accomplished by restricting θ to lie in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

In the following table we list trigonometric substitutions that are effective for the given radical expressions because of the specified trigonometric identities.

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \quad 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

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Trigonometric Substitution (6 of 8)

In each case the restriction on θ is imposed to ensure that the function that defines the substitution is one-to-one.

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Example 1

Evaluate $\int \frac{\sqrt{9 - x^2}}{x^2}\, dx$.

Solution:

Let $x = 3\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 3\cos\theta\, d\theta$ and

$$\begin{aligned}\sqrt{9 - x^2} &= \sqrt{9 - 9\sin^2\theta} \\ &= \sqrt{9\cos^2\theta} \\ &= 3|\cos\theta| \\ &= 3\cos\theta\end{aligned}$$

(Note that $\cos\theta \ge 0$ because $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.)

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Example 1 – *Solution* (1 of 4)

Thus the Inverse Substitution Rule gives

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{3\cos\theta}{9\sin^2\theta}3\cos\theta\,d\theta$$
$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$
$$= \int \cot^2\theta\,d\theta$$
$$= \int (\csc^2\theta - 1)\,d\theta$$
$$= -\cot\theta - \theta + C$$

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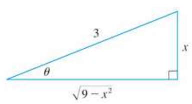
Example 1 – *Solution* (2 of 4)

Since this is an indefinite integral, we must return to the original variable *x*. This can be done either by using trigonometric identities to express cot θ in terms of $\sin\theta = \frac{x}{3}$ or by drawing a diagram, as in Figure 1, where θ is interpreted as an angle of a right triangle.



$\sin\theta = \frac{x}{3}$

Figure 1

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Example 1 – *Solution* (3 of 4)

Since $\sin\theta = \frac{x}{3}$, we label the opposite side and the hypotenuse as having lengths *x* and 3.

Then the Pythagorean Theorem gives the length of the adjacent side as $\sqrt{9-x^2}$, so we can simply read the value of cot θ from the figure:

$$\cot\theta = \frac{\sqrt{9-x^2}}{x}$$

(Although $\theta > 0$ in the diagram, this expression for cot θ is valid even when $\theta < 0$.)

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Example 1 – *Solution* (4 of 4)

Since $\sin\theta = \frac{x}{3}$, we have $\theta = \sin^{-1}\left(\frac{x}{3}\right)$ and so

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

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Example 2

Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution:
Solving the equation of the ellipse for *y*, we get

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2} \quad \text{or} \quad y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

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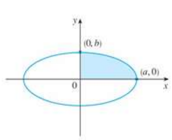
Example 2 – *Solution* (1 of 5)

Because the ellipse is symmetric with respect to both axes, the total area *A* is four times the area in the first quadrant (see Figure 2).



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Figure 2

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Example 2 – *Solution* (2 of 5)

The part of the ellipse in the first quadrant is given by the function

$$y = \frac{b}{a}\sqrt{a^2 - x^2} \qquad 0 \le x \le a$$

and so

$$\tfrac{1}{4}A = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\ dx$$

To evaluate this integral we substitute $x = a \sin \theta$. Then $dx = a \cos \theta\, d\theta$.

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Example 2 – *Solution* (3 of 5)

To change the limits of integration we note that when $x = 0$, $\sin \theta = 0$, so $\theta = 0$; when $x = a$, $\sin \theta = 1$, so $\theta = \frac{\pi}{2}$.

Also

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2 \theta} \\ &= \sqrt{a^2 \cos^2 \theta} \\ &= a|\cos \theta| \\ &= a \cos \theta\end{aligned}$$

since $0 \le \theta \le \frac{\pi}{2}$.

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Example 2 – *Solution* (4 of 5)

Therefore

$$\begin{aligned}A &= 4\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\ dx \\ &= 4\frac{b}{a}\int_0^{\pi/2} a\cos\theta \cdot a\cos\theta\ d\theta \\ &= 4ab\int_0^{\pi/2} \cos^2\theta\ d\theta \\ &= 4ab\int_0^{\pi/2} \tfrac{1}{2}(1 + \cos 2\theta)\ d\theta\end{aligned}$$

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Example 2 – *Solution* (5 of 5)

$$= 2ab[\theta + \tfrac{1}{2}\sin 2\theta]_0^{\pi/2}$$
$$= 2ab\left(\frac{\pi}{2} + 0 - 0\right)$$
$$= \pi ab$$

We have shown that the area of an ellipse with semiaxes a and b is πab.

In particular, taking $a = b = r$, we have proved the famous formula that the area of a circle with radius r is πr^2.

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Trigonometric Substitution (7 of 8)

Note:

Since the integral in Example 2 was a definite integral, we changed the limits of integration and did not have to convert back to the original variable x.

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Example 3

Find $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$.

Solution:

Let $x = 2\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 2\sec^2\theta\, d\theta$ and

$$\sqrt{x^2+4} = \sqrt{4(\tan^2\theta + 1)} = \sqrt{4\sec^2\theta}$$
$$= 2|\sec\theta|$$
$$= 2\sec\theta$$

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Example 3 – *Solution* (1 of 3)

So we have

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta \cdot 2\sec\theta}$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

To evaluate this trigonometric integral we put everything in terms of sin *θ* and cos *θ* :

$$\frac{\sec\theta}{\tan^2\theta} = \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin^2\theta}$$

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Example 3 – *Solution* (2 of 3)

Therefore, making the substitution $u = \sin\theta$, we have

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \frac{du}{u^2}$$

$$= \frac{1}{4}\left(-\frac{1}{u}\right) + C$$

$$= -\frac{1}{4\sin\theta} + C$$

$$= -\frac{\csc\theta}{4} + C$$

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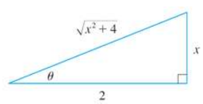
Example 3 – *Solution* (3 of 3)

We use Figure 3 to determine that $\csc\theta = \frac{\sqrt{x^2+4}}{x}$ and so

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = -\frac{\sqrt{x^2+4}}{4x} + C$$



$\tan\theta = \frac{x}{2}$

Figure 3

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Example 5

Evaluate $\int \frac{dx}{\sqrt{x^2 - a^2}}$, where $a > 0$.

Solution 1:

We let $x = a \sec \theta$, where $0 < \theta < \frac{\pi}{2}$ or $\pi < \theta < \frac{3\pi}{2}$.

Then $dx = a \sec \theta \tan \theta \, d\theta$ and

$$\begin{aligned}\sqrt{x^2 - a^2} &= \sqrt{a^2(\sec^2\theta - 1)} \\ &= \sqrt{a^2 \tan^2\theta} \\ &= a|\tan\theta| \\ &= a\tan\theta\end{aligned}$$

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Example 5 – *Solution 1* (1 of 3)

Therefore

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2 - a^2}} &= \int \frac{a\sec\theta\tan\theta}{a\tan\theta}\, d\theta \\ &= \int \sec\theta \, d\theta \\ &= \ln|\sec\theta + \tan\theta| + C\end{aligned}$$

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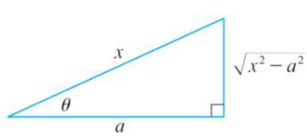
Example 5 – *Solution 1* (2 of 3)

The triangle in Figure 4 gives $\tan\theta = \frac{\sqrt{x^2 - a^2}}{a}$, so we have

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2 - a^2}} &= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + C \\ &= \ln\left|x + \sqrt{x^2 - a^2}\right| - \ln a + C\end{aligned}$$



$\sec\theta = \frac{x}{a}$

Figure 4

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Example 5 – *Solution* 1 (3 of 3)

Writing $C_1 = C - \ln a$, we have

1 $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right| + C_1$$

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Example 5 – *Solution* 2 (1 of 2)

For $x > 0$ the hyperbolic substitution $x = a \cosh t$ can also be used.

Using the identity $\cosh^2 y - \sinh^2 y = 1$, we have

$$\begin{aligned} \sqrt{x^2 - a^2} &= \sqrt{a^2(\cosh^2 t - 1)} \\ &= \sqrt{a^2 \sinh^2 t} \\ &= a \sinh t \end{aligned}$$

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Example 5 – *Solution* 2 (2 of 2)

Since $dx = a \sinh t\, dt$, we obtain

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2 - a^2}} &= \int \frac{a \sinh t\, dt}{a \sinh t} \\ &= \int dt \\ &= t + C \end{aligned}$$

Since $\cosh t = \frac{x}{a}$, we have $t = \cosh^{-1}\left(\frac{x}{a}\right)$ and

2 $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$$

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Trigonometric Substitution (8 of 8)

Note:

As Example 5 illustrates, hyperbolic substitutions can be used in place of trigonometric substitutions and sometimes they lead to simpler answers.

But we usually use trigonometric substitutions because trigonometric identities are more familiar than hyperbolic identities.

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Example 6

Find $\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\, dx$.

Solution:

First we note that $(4x^2+9)^{3/2} = (\sqrt{4x^2+9})^3$ so trigonometric substitution is appropriate.

Although $\sqrt{4x^2+9}$ is not quite one of the expressions in the table of trigonometric substitutions, it becomes one of them if we make the preliminary substitution $u = 2x$.

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Example 6 – *Solution* (1 of 3)

When we combine this with the tangent substitution, we have $x = \frac{3}{2}\tan\theta$, which gives $dx = \frac{3}{2}\sec^2\theta\, d\theta$ and

$$\sqrt{4x^2+9} = \sqrt{9\tan^2\theta+9}$$
$$= 3\sec\theta$$

When $x = 0$, $\tan\theta = 0$, so $\theta = 0$; when $x = \frac{3\sqrt{3}}{2}$, $\tan\theta = \sqrt{3}$, so $\theta = \frac{\pi}{3}$.

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\, dx = \int_0^{\pi/3} \frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta} \frac{3}{2}\sec^2\theta\, d\theta$$

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Example 6 – *Solution* (2 of 3)

$$= \tfrac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\, d\theta$$

$$= \tfrac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta}\, d\theta$$

$$= \tfrac{3}{16}\int_0^{\pi/3} \frac{1-\cos^2\theta}{\cos^2\theta}\sin\theta\, d\theta$$

Now we substitute $u = \cos\theta$ so that $du = -\sin\theta\, d\theta$. When $\theta = 0$, $u = 1$; when $\theta = \frac{\pi}{3}$, $u = \frac{1}{2}$.

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Example 6 – *Solution* (3 of 3)

Therefore

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\, dx = -\tfrac{3}{16}\int_1^{1/2} \frac{1-u^2}{u^2}\, du$$

$$= \tfrac{3}{16}\int_1^{1/2} (1-u^{-2})\, du$$

$$= \tfrac{3}{16}\left[u + \frac{1}{u}\right]_1^{1/2}$$

$$= \tfrac{3}{16}\left[\left(\tfrac{1}{2}+2\right)-(1+1)\right]$$

$$= \frac{3}{32}$$

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