7.2 Trigonometric Integrals

Find $\int \sin^5 x \cos^2 x \, dx$.

$u = \sin x$ | $u = \cos x$

$du = \cos x$ | $du = -\sin x$

$\sin^2 x + \cos^2 x = 1$

Find $\int \sin^5 x \cos^2 x \; dx$.

$$\cos^2 x = 1 - \sin^2 x$$

$$\int \cos^2 x \boxed{\sin^4 x} \sin x \, dx$$

$$\int \cos^2 x \, (1-\cos^2 x)^2 \sin x \, dx$$

$$u = \cos x$$

$$du = -\sin x \rightarrow -du = \sin x \, dx$$

Find $\int \sin^5 x \cos^2 x \; dx$.

$$\cos^2 x = 1 - \sin^2 x$$

$$\int \cos^2 x (1-\cos^2 x)^2 \boxed{\sin x \, dx}$$

$$\int u^2(1-u^2)^2(-du) \qquad u = \cos x$$

$$-\int u^2 - 2u^4 + u^6 \, du \qquad -du = \sin x \, dx$$

$$-\left[\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right] + C$$

$$\boxed{-\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x + \frac{1}{7}\cos^7 x}$$

$$u^2(1-u^2)^2 \longleftrightarrow (1-u^2)(1-u^2)$$

$$u^2(1-2u^2+u^4)$$

$$u^2-2u^4+u^6$$

$$\int \sin^2 x \cos^3 x \, dx$$

$$u = \sin x$$
$$du = \cos x$$

$$\int \sin^2 x \boxed{\cos^2 x} \cos x \, dx$$

$$\int \sin^2 x (1 - \sin^2 x) \boxed{\cos x \, dx} \leftarrow du$$

$$\int u^2 (1 - u^2) \, du$$

$$\int u^2 - u^4 \, du = \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \boxed{1/3 \sin^3 x - \frac{1}{5} \sin^5 x + C}$$

Evaluate $\int_0^{\pi} \sin^2 x \, dx$.

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\sin x \cos x = \frac{1}{2}\sin 2x$

$$\int_0^\pi \sin^2 x \, dx \qquad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\int_0^\pi \frac{1}{2}(1 - \cos 2x)\, dx$$

$$\boxed{\begin{aligned} u &= 2x \\ du &= 2\, dx \\ \tfrac{1}{2} du &= dx \end{aligned}}$$

$$\frac{1}{2}\int_0^\pi 1 - \cos 2x \, dx$$

$$\frac{1}{2}\left[\int_0^\pi 1 - \frac{1}{2}\int_0^\pi \cos u \, du\right]$$

$$\frac{1}{2}\left[x - \frac{1}{2}(-\sin 2x)\right]_0^\pi$$

$$\frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi}$$

$$\frac{1}{2}\left[(\pi)\right] = \frac{1}{2}\pi$$

$\sin 2\pi = 0$

$\sin 0 = 0$

$$\int \cos^4(2t)\,dt$$

$x = 2t$

$dx = 2\,dt \rightarrow \frac{1}{2}dx = dt$

$$\frac{1}{2}\int \cos^4 x \; dx$$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$$\frac{1}{2}\int \left[\frac{1}{2}(1+\cos 2x)\right]^2 dx$$

$$\frac{1}{8}\int 1 + 2\cos 2x + \cos^2 2x \; dx$$

$$\frac{1}{8}\int 1+2\cos 2x+\cos^2 2x \quad dx \qquad x=2t$$

$$\frac{1}{8}\int 1+2\cos 2x+\frac{1}{2}\left(1+\cos 4x\right) dx$$

$$u=4x$$

$$\frac{1}{8}\int \frac{3}{2}+2\cos 2x+\frac{1}{2}\cos 4x \quad dx \qquad \frac{1}{4}du=dx$$

$$\frac{1}{8}\left[\frac{3}{2}x+2\left(\frac{1}{2}\right)\left(-\sin 2x\right)+\frac{1}{2}\left(\frac{1}{4}\right)\left(-\sin 4x\right)\right]$$

$$\frac{3}{16}x-\frac{1}{8}\sin 2x-\frac{1}{64}\sin 4x \qquad x=2t$$

$$3/16\,(2t) \qquad \boxed{\frac{3}{8}t-\frac{1}{8}\sin 4t-\frac{1}{64}\sin 8t+C}$$

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\begin{aligned}\int \sin^m x \cos^{2k+1} x \, dx &= \int \sin^m x (\cos^2 x)^k \cos x \, dx \\ &= \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx\end{aligned}$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\begin{aligned}\int \sin^{2k+1} x \cos^n x \, dx &= \int (\sin^2 x)^k \cos^n x \sin x \, dx \\ &= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx\end{aligned}$$

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \quad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2} \sin 2x$$

Evaluate $\int \tan^6 x \sec^4 x \, dx$.

$\tan^2 x = \sec^2 x - 1$

$$\boxed{\begin{aligned} u &= \sec x \\ du &= \sec x \tan x \, dx \end{aligned}}$$

$$\boxed{\begin{aligned} u &= \tan x \\ du &= \sec^2 x \, dx \end{aligned}}$$

$$\int \tan^6 x \sec^4 x \, dx$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\int \tan^6 x \sec^2 x \sec^2 x \, dX$$

$$u = \tan x$$

$$du = \sec^2 x \, dx$$

$$\int \tan^6 x (\tan^2 x + 1) \boxed{\sec^2 x \, dx}$$

$$\int u^6 (u^2 + 1) \, du$$

$$\int u^8 + u^6 \, du$$

$$\frac{u^9}{9} + \frac{u^7}{7} + C$$

$$\boxed{\frac{1}{9} \tan^9 x + \frac{1}{7} \tan^7 x + C}$$

$$\int \tan^5 x \sec^7 x \, dx$$

$$u = \sec x$$

$$du = \tan x \sec x \, dx$$

$$\tan^2 x = \sec^2 x - 1$$

$$\int \tan^4 x \sec^6 x \tan x \sec x \, dx$$

$$\int (\sec^2 x - 1)^2 \sec^6 x \tan x \sec x \, dx$$

$$\int (u^2 - 1)^2 u^6 \, du \qquad \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} + C$$

$$\int u^{10} - 2u^8 + u^6 \, du \qquad \boxed{1/11 \sec^{11} x - \frac{2}{9} \sec^9 x + \frac{1}{7} \sec^7 x + C}$$

Strategy for Evaluating $\int \tan^m x \sec^n x\,dx$

(a) If the power of secant is even ($n = 2k$, $k \geq 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\begin{aligned}\int \tan^m x \sec^{2k} x\,dx &= \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x\,dx \\ &= \int \tan^m x (1+\tan^2 x)^{k-1} \sec^2 x\,dx\end{aligned}$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\begin{aligned}\int \tan^{2k+1} x \sec^n x\,dx &= \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x\,dx \\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x\,dx\end{aligned}$$

Then substitute $u = \sec x$.

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$$

Find $\int \tan^3 x\, dx$.

$\tan^2 x = \sec^2 x - 1$

$\int \tan x \tan^2 x\, dx$

$u = \tan x$

$du = \sec^2 x\, dx$

$\int \tan x (\sec^2 x - 1)\, dx$

$\int \tan x \sec^2 x\, dx - \int \tan x\, dx$

$\int u\, du$

$\boxed{\frac{1}{2}\tan^2 x - \ln|\sec x| + C}$

$$\int \tan x \sec^2 x \, dx \qquad u = \tan x$$

$$du = \sec^2 x$$

$$\int u \, du$$

$$\frac{u^2}{2} + C \rightarrow \frac{\tan^2 x}{2} + C$$

Finally, we can make use of another set of trigonometric identities:

2 To evaluate the integrals (a) $\int \sin mx \cos nx \; dx$, (b) $\int \sin mx \sin nx \; dx$, or (c) $\int \cos mx \cos nx \; dx$, use the corresponding identity:

(a) $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

(b) $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

(c) $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Evaluate $\int \sin 4x \cos 5x \, dx$.