In finding the area of a circle or an ellipse, an integral of the form $\int \sqrt{a^2 - x^2}\, dx$ arises, where $a > 0$. If it were $\int x\sqrt{a^2 - x^2}\, dx$, the substitution $u = a^2 - x^2$ would be effective but, as it stands, $\int \sqrt{a^2 - x^2}\, dx$ is more difficult. If we change the variable from $x$ to $\theta$ by the substitution $x = a \sin \theta$, then the identity $1 - \sin^2\theta = \cos^2\theta$ allows us to get rid of the root sign because

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2 \cos^2\theta} = a\left|\cos \theta\right|$$

Trig Substitution

$x = a\sin\theta$

$\sin^2\theta + \cos^2\theta = 1$

$$\boxed{\sqrt{a^2 - x^2}}$$

$$\sqrt{a^2 - (a\sin\theta)^2}$$

$$\sqrt{a^2 - a^2\sin^2\theta}$$

$$\sqrt{a^2(1 - \sin^2\theta)}$$

$$\sqrt{a^2}\sqrt{1 - \sin^2\theta}$$

$$a\sqrt{\cos^2\theta}$$

$$\boxed{a\left|\cos\theta\right|}$$

## Table of Trigonometric Substitutions

| Expression | Substitution | Identity |
|---|---|---|
| $\sqrt{a^2 - x^2}$ | $x = a \sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$ | $1 - \sin^2\theta = \cos^2\theta$ |
| $\sqrt{a^2 + x^2}$ | $x = a \tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$ | $1 + \tan^2\theta = \sec^2\theta$ |
| $\sqrt{x^2 - a^2}$ | $x = a \sec\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$ | $\sec^2\theta - 1 = \tan^2\theta$ |

**EXAMPLE 1** Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$.

$x = 3\sin\theta$

$dx = \boxed{3\cos\theta \; d\theta}$

$$\int \frac{\sqrt{9-(3\sin\theta)^2}}{(3\sin\theta)^2}\, 3\cos\theta \, d\theta$$

$$\sqrt{9-(3\sin\theta)^2} \qquad \int \frac{3\cos\theta}{9\sin^2\theta}\cdot 3\cos\theta \, d\theta$$

$$\sqrt{9-9\sin^2\theta} \qquad \int \frac{9\cos^2\theta}{9\sin^2\theta} \, d\theta$$

$$\sqrt{9(1-\sin^2\theta)} \qquad \int \cot^2\theta \, d\theta$$

$$\sqrt{9(\cos^2\theta)} \qquad \int \csc^2\theta - 1 \, d\theta$$

$$3\cos\theta \qquad -\cot\theta - \theta + C$$

$$-\cot\theta - \theta + C$$

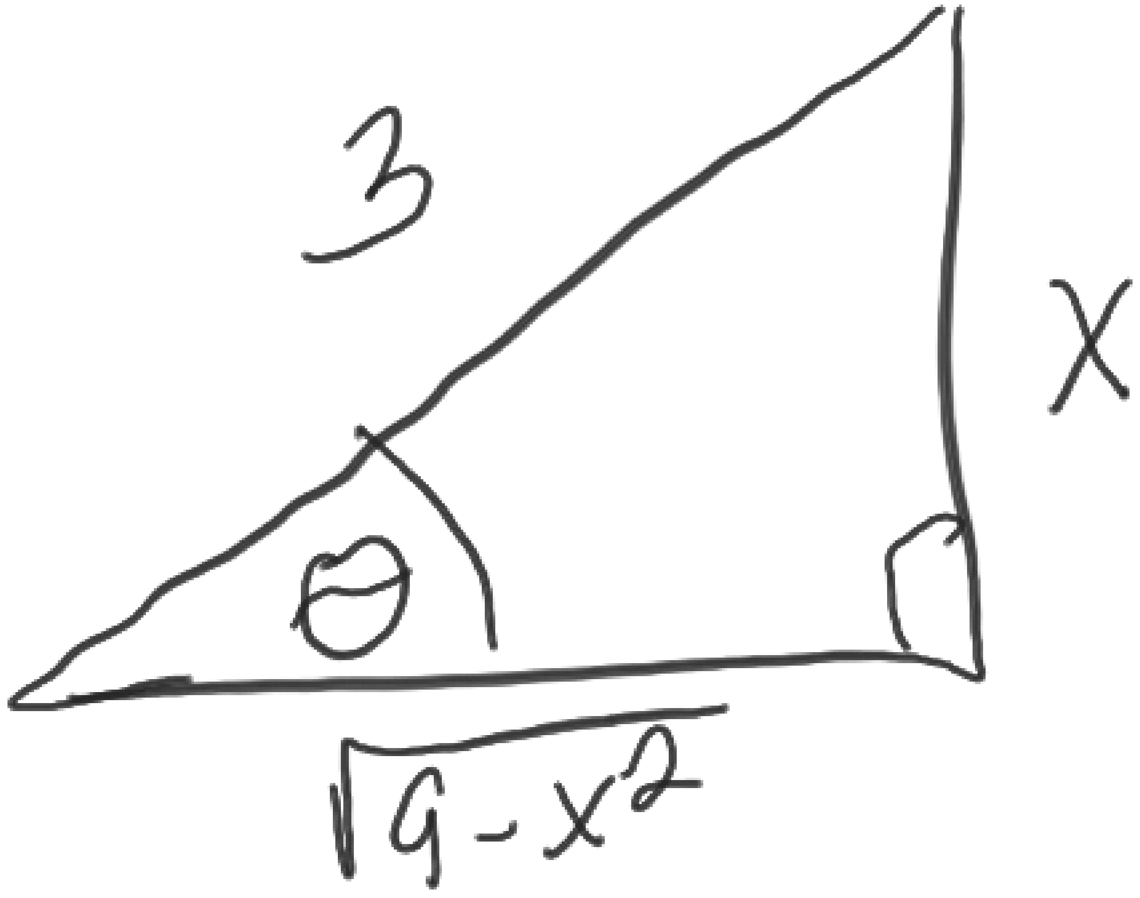
$$\boxed{-\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\frac{x}{3} + C}$$

$$\left(\frac{x}{3} = \sin\theta\right)\sin^{-1}$$

$$\sin^{-1}\frac{x}{3} = \theta$$

$$\frac{x}{3} = \frac{3\sin\theta}{3}$$

$$\frac{x}{3} = \sin\theta$$

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Find $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx.$

$$x = 2\tan\theta$$

$$dx = 2\sec^2\theta\,d\theta$$

$$\int \frac{2\sec^2\theta}{(2\tan\theta)^2\sqrt{(2\tan\theta)^2+4}}\,d\theta$$

$$\sqrt{(2\tan\theta)^2+4}$$

$$\sqrt{4\tan^2\theta+4}$$

$$\sqrt{4(\tan^2\theta+1)}$$

$$\sqrt{4\sec^2\theta}$$

$$2\sec\theta$$

$$\int \frac{2\sec^2\theta}{4\tan^2\theta \cdot 2\sec\theta}\,d\theta = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$\frac{1}{4}\int \boxed{\frac{\cos\theta}{\sin^2\theta}\,d\theta}$$

$$u = \sin\theta$$

$$du = \boxed{\cos\theta\,d\theta}$$

$$\frac{1}{4}\int \frac{1}{u^2}\,du$$

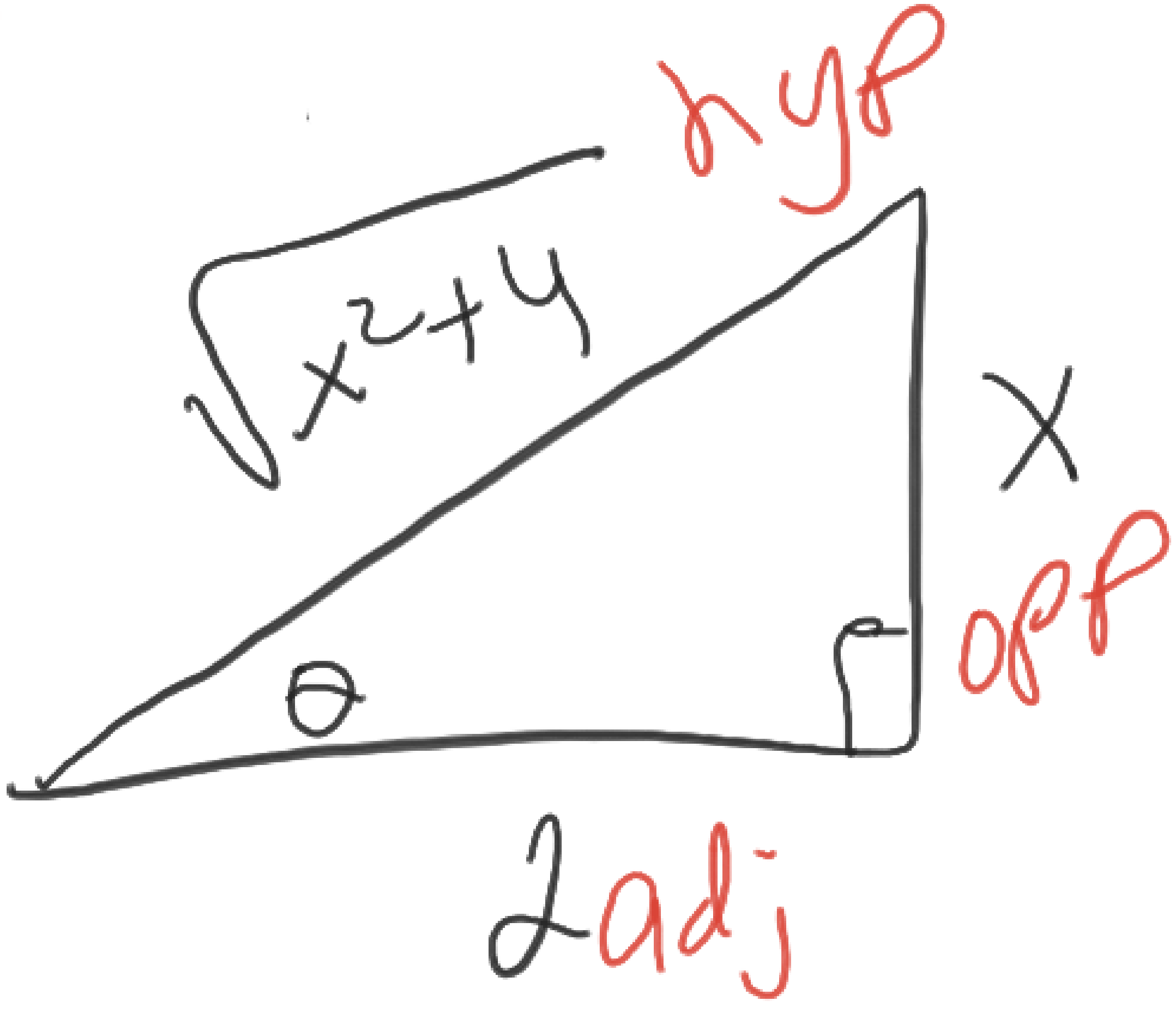
$$\frac{1}{4}\left(\frac{-1}{u}\right)+C = \frac{-1}{4\sin\theta}+C$$

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = -\frac{1}{4}\boxed{\frac{\sqrt{x^2+4}}{x}} = \boxed{\frac{-\sqrt{x^2+4}}{4x}}$$

$$\frac{-1}{4\sin\theta} = -\frac{1}{4}\boxed{\csc\theta} = \frac{hyp}{opp}$$

$$x = 2\tan\theta$$

$$\tan\theta = \frac{x}{2}$$



Find $\int \frac{x}{\sqrt{x^2 + 4}}\,dx$.

$u = x^2 + 4$

$du = 2x\,dx$

$\frac{1}{2}du = \boxed{x\,dx}$

$\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$

$\frac{1}{2}\int u^{-1/2}\,du$

$\frac{1}{2} \cdot 2u^{1/2} + C \rightarrow \sqrt{u} + C = \boxed{\sqrt{x^2+4} + C}$

Find $\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx.$ $= \int_{0}^{\frac{3\sqrt{3}}{2}} \frac{x^3}{\left(\sqrt{4x^2+9}\right)^3}\,dx$

$x = \frac{3}{2}\tan\theta$

$dx = \frac{3}{2}\sec^2\theta\,d\theta$

$\sqrt{a^2+x^2}$

$a^2 = 9 \quad x^2 = 4x^2$

$a = 3 \quad x = (2x)^2$

$\frac{2x}{2} = \frac{3\tan\theta}{2}$

$0 = \frac{3}{2}\tan\theta$

$0 = \tan\theta$

$0 = \theta$

$\frac{3\sqrt{3}}{2} = \frac{3}{2}\tan\theta$

$\sqrt{3} = \tan\theta$

$\frac{\pi}{3} = \theta$

$$\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{3/2}}\,dx = \int_0^{\pi/3} \frac{\left(\frac{3}{2}\tan\theta\right)^3}{\left(\sqrt{4\left(\frac{3}{2}\tan\theta\right)^2+9}\right)^3} \cdot \frac{3}{2}\sec^2\theta\,d\theta$$

$$\boxed{\begin{array}{c} \sqrt{9(\tan^2\theta+1)} \\ \sqrt{9\sec^2\theta} \\ 3\sec\theta \end{array}}$$

$$\int_0^{\pi/3} \frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta} \cdot \frac{3}{2}\sec^2\theta\,d\theta$$

$$\frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\,d\theta$$

$$\frac{3}{16}\int_0^{\pi/3} \boxed{\frac{\tan^3\theta}{\sec\theta}}\, d\theta = \frac{3}{16}\int_0^{\pi/3} \boxed{\frac{\sin^3\theta}{\cos^2\theta}}\, d\theta$$

$\sin^2 \cdot \sin$

$$\frac{3}{16}\int_0^{\pi/3} \frac{\boxed{(1-\cos^2\theta)}}{\cos^2\theta}\sin\theta\, d\theta$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$u = \cos 0 = 1$

$u = \cos \pi/3 = 1/2$

$$\frac{-3}{16}\int_1^{1/2} \frac{1-u^2}{u^2}\, du = \frac{3}{16}\int_{1/2}^{1} u^{-2} - 1\, du$$

$$1/u^2 - u^2/u^2$$

$$\frac{3}{16}\int_{1/2}^{1} u^{-2}-1\,du = \frac{3}{16}\left[-\frac{1}{u}-u\right]_{1/2}^{1}$$

$$\frac{3}{16}\left[(-1-1)-\left(-2-\frac{1}{2}\right)\right]$$

$$3/16\,(-2+2+1/2) = 3/32$$