Chapter 10

Parametric Equations and Polar Coordinates

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10.2 Calculus with Parametric Curves

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Tangents

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Tangents (1 of 3)

Suppose *f* and *g* are differentiable functions and we want to find the tangent line at a point on the parametric curve $x = f(t)$, $y = g(t)$, where *y* is also a differentiable function of *x*.

Then the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

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Tangents (2 of 3)

If $\frac{dx}{dt} \neq 0$, we can solve for $\frac{dy}{dx}$:

$$\textbf{1} \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if } \frac{dx}{dt} \neq 0$$

Equation 1 (which you can remember by thinking of canceling the *dt*'s) enables us to find the slope $\frac{dy}{dx}$ of the tangent to a parametric curve without having to eliminate the parameter *t*.

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Tangents (3 of 3)

We see from (1) that the curve has a horizontal tangent when $\frac{dy}{dt} = 0$ (provided that $\frac{dx}{dt} \neq 0$) and it has a vertical tangent when $\frac{dx}{dt} = 0$ (provided that $\frac{dy}{dt} \neq 0$).

This information is useful for sketching parametric curves.

It is also useful to consider $\frac{d^2y}{dx^2}$. This can be found by replacing *y* by $\frac{dy}{dx}$ in Equation 1:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

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Example 1

A curve C is defined by the parametric equations $x = t^2,\ y = t^3 - 3t$.

(a) Show that C has two tangents at the point (3, 0) and find their equations.
(b) Find the points on C where the tangent is horizontal or vertical.
(c) Determine where the curve is concave upward or downward.
(d) Sketch the curve.

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Example 1 – *Solution* (1 of 5)

(a) Notice that $y = t^3 - 3t = t\left(t^2 - 3\right) = 0$ when $t = 0$ or $t = \pm\sqrt{3}$. Therefore the point (3, 0) on C arises from two values of the parameter, $t = \sqrt{3}$ and $t = -\sqrt{3}$. This indicates that C crosses itself at (3, 0).

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Example 1 – *Solution* (2 of 5)

Since

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t}$$

$$= \frac{3}{2}\left(t - \frac{1}{t}\right)$$

the slope of the tangent when $t = \pm\sqrt{3}$ is $\frac{dy}{dx} = \pm\frac{6}{\left(2\sqrt{3}\right)} = \pm\sqrt{3}$, so the equations of the tangents at (3, 0) are

$$y = \sqrt{3}\left(x - 3\right) \quad \text{and} \quad y = -\sqrt{3}\left(x - 3\right)$$

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Example 1 – *Solution* (3 of 5)

(b) C has a horizontal tangent when $\frac{dy}{dx} = 0$, that is, when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.

Since $\frac{dy}{dt} = 3t^2 - 3$, this happens when $t^2 = 1$, that is, $t = \pm 1$.

The corresponding points on C are (1, −2) and (1, 2).

C has a vertical tangent when $\frac{dx}{dt} = 2t = 0$, that is, $t = 0$. (Note that $\frac{dy}{dt} \neq 0$ there.)

The corresponding point on C is (0, 0).

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Example 1 – *Solution* (4 of 5)

(c) To determine concavity we calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{3}{2}\left(1 + \frac{1}{t^2}\right)}{2t}$$

$$= \frac{3\left(t^2 + 1\right)}{4t^3}$$

Thus the curve is concave upward when $t > 0$ and concave downward when $t < 0$.

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Example 1 – *Solution* (5 of 5)

(d) Using the information from parts (b) and (c), we sketch C in Figure 1.

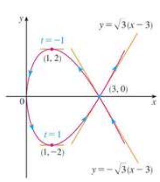

Figure 1

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Areas

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Areas

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)\,dx$, where $F(x) \geq 0$.

If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t) f'(t)\,dt \qquad \left[\text{or} \int_\beta^\alpha g(t) f'(t)\,dt\right]$$

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Example 3

Find the area under one arch of the cycloid

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta)$$

(See Figure 3.)

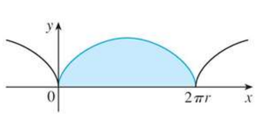

Figure 3

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Example 3 – *Solution* (1 of 2)

One arch of the cycloid is given by $0 \le \theta \le 2\pi$.

Using the Substitution Rule with $y = r(1 - \cos\theta)$ and $dx = r(1 - \cos\theta)d\theta$, we have

$$A = \int_0^{2\pi r} y\,dx = \int_0^{2\pi} r(1-\cos\theta)\,r(1-\cos\theta)\,d\theta$$
$$= r^2\int_0^{2\pi}(1-\cos\theta)^2\,d\theta$$
$$= r^2\int_0^{2\pi}\left(1-2\cos\theta+\cos^2\theta\right)d\theta$$

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Example 3 – *Solution* (2 of 2)

$$= r^2\int_0^{2\pi}\left[1-2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right]d\theta$$
$$= r^2\left[\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{2\pi}$$
$$= r^2\left(\frac{3}{2}\cdot 2\pi\right) = 3\pi r^2$$

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Arc Length

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Arc Length (1 of 9)

We already know how to find the length L of a curve C given in the form $y = F(x)$, $a \le x \le b$.

If F' is continuous, then

$$\textbf{2} \quad L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Suppose that C can also be described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \le t \le \beta$, where $\frac{dx}{dt} = f'(t) > 0$.

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Arc Length (2 of 9)

This means that C is traversed once, from left to right, as t increases from α to β and $f(\alpha) = a$, $f(\beta) = b$.

Putting Formula 1 into Formula 2 and using the Substitution Rule, we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\, \frac{dx}{dt}\, dt$$

Since $\frac{dx}{dt} > 0$, we have

$$\textbf{3} \quad L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

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Arc Length (3 of 9)

Even if C can't be expressed in the form $y = F(x)$, Formula 3 is still valid but we obtain it by polygonal approximations.

We divide the parameter interval $[\alpha, \beta]$ into n subintervals of equal width Δt.

If $t_0, t_1, t_2, \ldots, t_n$ are the endpoints of these subintervals, then $x_i = f(t_i)$ and $y_i = g(t_i)$ are the coordinates of points $P_i(x_i, y_i)$ that lie on C and the polygon with vertices $P_0, P_1, \ldots, P_n$ approximates C. (See Figure 4.)

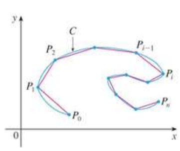

Figure 4

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Arc Length (4 of 9)

We define the length L of C to be the limit of the lengths of these approximating polygons as $n \to \infty$:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

The Mean Value Theorem, when applied to f on the interval $[t_{i-1}, t_i]$, gives a number t_i^* in (t_{i-1}, t_i) such that

$$f(t_i) - f(t_{i-1}) = f'(t_i^*)(t_i - t_{i-1})$$

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Arc Length (5 of 9)

If we let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, this equation becomes

$$\Delta x_i = f'(t_i^*)\Delta t$$

Similarly, when applied to g, the Mean Value Theorem gives a number t_i^{**} in (t_{i-1}, t_i) such that

$$\Delta y_i = g'(t_i^{**})\Delta t$$

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Arc Length (6 of 9)

Therefore

$$|P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{\left[f'(t_i^*)\Delta t\right]^2 + \left[g'(t_i^{**})\Delta t\right]^2}$$
$$= \sqrt{\left[f'(t_i^*)^2\right] + \left[g'(t_i^{**})\right]^2}\,\Delta t$$

and so

$$\textbf{4} \quad L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\left[f'(t_i^*)\right]^2 + \left[g'(t_i^{**})\right]^2}\,\Delta t$$

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Arc Length (7 of 9)

The sum in (4) resembles a Riemann sum for the function $\sqrt{\left[f'(t)^2 + g'(t)\right]^2}$ but it is not exactly a Riemann sum because $t_i^* \neq t_i^{**}$ in general.

Nevertheless, if f' and g' are continuous, it can be shown that the limit in (4) is the same as if t_i^* and t_i^{**} were equal, namely,

$$L = \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

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Arc Length (8 of 9)

Thus, using Leibniz notation, we have the following result, which has the same form as Formula 3.

5 Theorem If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$). where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Notice that the formula in Theorem 5 is consistent with the general formulas $L = \int ds$ and $(ds)^2 = (dx)^2 + (dy)^2$.

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Arc Length (9 of 9)

Notice that the integral gives twice the arc length of the circle because as t increases from 0 to 2π, the point ($\sin 2t$, $\cos 2t$) traverses the circle twice.

In general, when finding the length of a curve C from a parametric representation, we have to be careful to ensure that C is traversed only once as t increases from α to β.

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Example 5

Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.

Solution:

From Example 3 we see that one arch is described by the parameter interval $0 \le \theta \le 2\pi$.

Since

$$\frac{dx}{d\theta} = r(1-\cos\theta) \quad \text{and} \quad \frac{dy}{d\theta} = r\sin\theta$$

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Example 5 – *Solution* (1 of 3)

We have

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{r^2(1-2\cos\theta+\cos^2\theta+\sin^2\theta)}\, d\theta \\ &= r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta \end{aligned}$$

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Example 5 – *Solution* (2 of 3)

To evaluate this integral we use the identity $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ with $\theta = 2x$, which gives $1-\cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$.

Since $0 \le \theta \le 2\pi$, we have $0 \le \frac{\theta}{2} \le \pi$ and so $\sin\left(\frac{\theta}{2}\right) \ge 0$.

Therefore

$$\begin{aligned} \sqrt{2(1-\cos\theta)} &= \sqrt{4\sin^2\left(\frac{\theta}{2}\right)} \\ &= 2\left|\sin\left(\frac{\theta}{2}\right)\right| \\ &= 2\sin\left(\frac{\theta}{2}\right) \end{aligned}$$

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Example 5 – *Solution* (3 of 3)

and so

$$L = 2r\int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) d\theta$$
$$= 2r\left[-2\cos\left(\frac{\theta}{2}\right)\right]_0^{2\pi}$$
$$= 2r[2+2]$$
$$= 8r$$

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Surface Area

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Surface Area

Suppose the curve *c* given by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$, where f', g' are continuous, $g(t) \ge 0$, is rotated about the *x*-axis. If *C* is traversed exactly once as *t* increases from α to β, then the area of the resulting surface is given by

6 $$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The general symbolic formulas $S = \int 2\pi y\, ds$ and $S = \int 2\pi x\, ds$ are still valid, but for parametric curves we use

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

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Example 6

Show that the surface area of a sphere of radius r is $4\pi r^2$.

Solution:

The sphere is obtained by rotating the semicircle

$$x = r\cos t \qquad y = r\sin t \qquad 0 \le t \le \pi$$

about the x-axis.

Therefore, from Formula 6, we get

$$S = \int_0^{\pi} 2\pi r \sin t \sqrt{(-r \sin t)^2 + (r \cos t)^2}\, dt$$

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Example 6 – *Solution*

$$= 2\pi \int_0^{\pi} r \sin t \sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt$$

$$= 2\pi \int_0^{\pi} r \sin t \cdot r\, dt$$

$$= 2\pi r^2 \int_0^{\pi} \sin t\, dt$$

$$= 2\pi r^2 (-\cos t)\Big]_0^{\pi}$$

$$= 4\pi r^2$$

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