Chapter 10

Parametric Equations and Polar Coordinates

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10.4 Areas and Lengths in Polar Coordinates

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Areas and Lengths in Polar Coordinates (1 of 7)

In this section we develop the formula for the area of a region whose boundary is given by a polar equation. We need to use the formula for the area of a sector of a circle:

$$1 \quad A = \frac{1}{2} r^2 \theta$$

where, as in Figure 1, r is the radius and θ is the radian measure of the central angle.

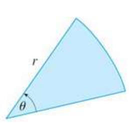

Figure 1

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Areas and Lengths in Polar Coordinates (2 of 7)

Formula 1 follows from the fact that the area of a sector is proportional to its central angle:

$$A = \left(\frac{\theta}{2\pi}\right)\pi r^2 = \frac{1}{2}r^2\theta.$$

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Areas and Lengths in Polar Coordinates (3 of 7)

Let $\mathfrak{R}$ be the region, illustrated in Figure 2, bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$, where f is a positive continuous function and where $0 < b - a \leq 2\pi$.

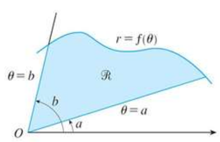

Figure 2

We divide the interval $[a, b]$ into subintervals with endpoints $\theta_0, \theta_1, \theta_2, \ldots, \theta_n$ and equal width $\Delta\theta$.

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Areas and Lengths in Polar Coordinates (4 of 7)

The rays $\theta = \theta_i$ then divide $\mathfrak{R}$ into n smaller regions with central angle $\Delta\theta = \theta_i - \theta_{i-1}$. If we choose θ^*_i in the ith subinterval $[\theta_{i-1}, \theta_i]$, then the area ΔA_i of the ith region is approximated by the area of the sector of a circle with central angle $\Delta\theta$ and radius $f(\theta^*_i)$. (See Figure 3.)

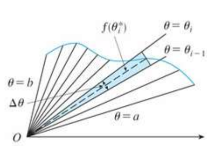

Figure 3

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Areas and Lengths in Polar Coordinates (5 of 7)

Thus from Formula 1 we have

$$\Delta A_i \approx \tfrac{1}{2}\left[f(\theta_i^*)\right]^2 \Delta\theta$$

and so an approximation to the total area A of $\mathfrak{R}$ is

$$\textbf{2} \quad A \approx \sum_{i=1}^{n} \tfrac{1}{2}\left[f(\theta_i^*)\right]^2 \Delta\theta$$

It appears from Figure 3 that the approximation in (2) improves as $n \to \infty$.

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Areas and Lengths in Polar Coordinates (6 of 7)

But the sums in (2) are Riemann sums for the function $g(\theta) = \tfrac{1}{2}[f(\theta)]^2$, so

$$\lim_{n\to\infty} \sum_{i=1}^{n} \tfrac{1}{2}[f(\theta_i^*)]^2 \Delta\theta = \int_a^b \tfrac{1}{2}[f(\theta)]^2 \, d\theta$$

It therefore appears plausible that the formula for the area A of the polar region $\mathfrak{R}$ is

$$\textbf{3} \quad A = \int_a^b \tfrac{1}{2}[f(\theta)]^2 d\theta$$

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Areas and Lengths in Polar Coordinates (7 of 7)

Formula 3 is often written as

$$\textbf{4} \quad A = \int_a^b \tfrac{1}{2} r^2 \, d\theta$$

with the understanding that $r = f(\theta)$. Note the similarity between Formulas 1 and 4.

When we apply Formula 3 or 4 it is helpful to think of the area as being swept out by a rotating ray through O that starts with angle a and ends with angle b.

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Example 1

Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Solution:

Notice from Figure 4 that the region enclosed by the right loop is swept out by a ray that rotates from $\theta = -\frac{\pi}{4}$ to $\theta = \frac{\pi}{4}$.

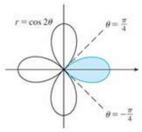

Figure 4

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Example 1 – *Solution* (1 of 2)

Therefore Formula 4 gives

$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tfrac{1}{2} r^2 \, d\theta$$

$$= \tfrac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta$$

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Example 1 – *Solution* (2 of 2)

$$A = \int_0^{\frac{\pi}{4}} \tfrac{1}{2}(1 + \cos 4\theta) \, d\theta$$

$$= \tfrac{1}{2}\Big[\theta + \tfrac{1}{4}\sin 4\theta\Big]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{8}$$

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Arc Length

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Arc Length (1 of 3)

To find the length of a polar curve $r = f(\theta)$, $a \le \theta \le b$, we regard θ as a parameter and write the parametric equations of the curve as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Using the Product Rule and differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

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Arc Length (2 of 3)

So, using $\cos^2\theta + \sin^2\theta = 1$, we have

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta$$

$$+\left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

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Arc Length (3 of 3)

Assuming that f' is continuous, we can write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Therefore the length of a curve with polar equation $r = f(\theta)$, $a \le \theta \le b$, is

$$5 \quad L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

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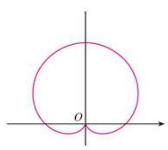
Example 4

Find the length of the cardioid $r = 1 + \sin\theta$.

Solution:

The cardioid is shown in Figure 8.



$r = 1 + \sin\theta$

Figure 8

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Example 4 – *Solution* (1 of 2)

$$5 \quad L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Its full length is given by the parameter interval $0 \le \theta \le 2\pi$, so Formula 5 gives

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{(1 + \sin\theta)^2 + \cos^2\theta}\, d\theta$$

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Example 4 – *Solution* (2 of 2)

$$= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta$$

We could evaluate this integral by multiplying and dividing the integrand by $\sqrt{2 - 2\sin\theta}$, or we could use a computer algebra system.
In any event, we find that the length of the cardioid is $L = 8$.

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