10.1 Curves Defined by Parametric Equations

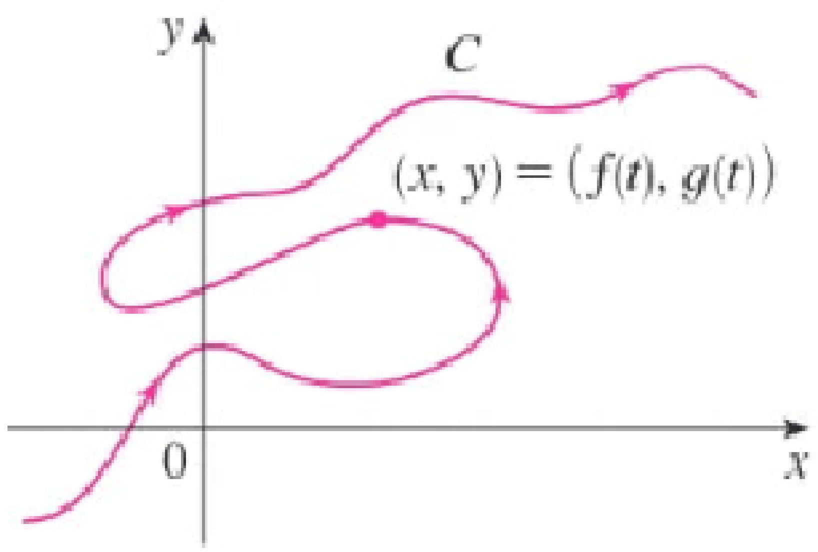


FIGURE 1

Imagine that a particle moves along the curve C shown in Figure 1. It is impossible to describe C by an equation of the form $y = f(x)$ because C fails the Vertical Line Test. But the x- and y-coordinates of the particle are functions of time and so we can write $x = f(t)$ and $y = g(t)$. Such a pair of equations is often a convenient way of describing a curve and gives rise to the following definition.

Suppose that x and y are both given as functions of a third variable t (called a **parameter**) by the equations

$$x = f(t) \qquad y = g(t)$$

(called **parametric equations**). Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a **parametric curve.** The parameter t does not necessarily represent time and, in fact, we could use a letter other than t for the parameter. But in many applications of parametric curves, t does denote time and therefore we can interpret $(x, y) = (f(t), g(t))$ as the position of a particle at time t.

EXAMPLE 1 Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1$$

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5

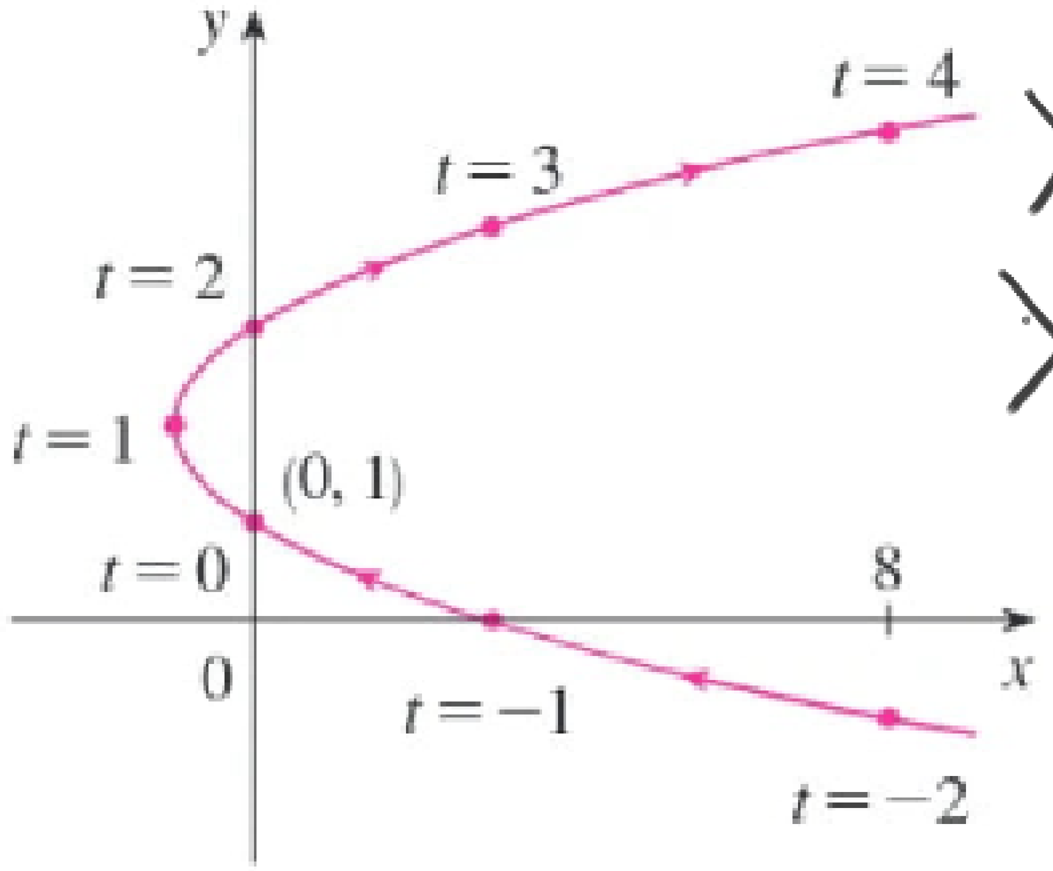


FIGURE 2

$t = -2$

$x = (-2)^2 - 2(-2)$

$x = 8$

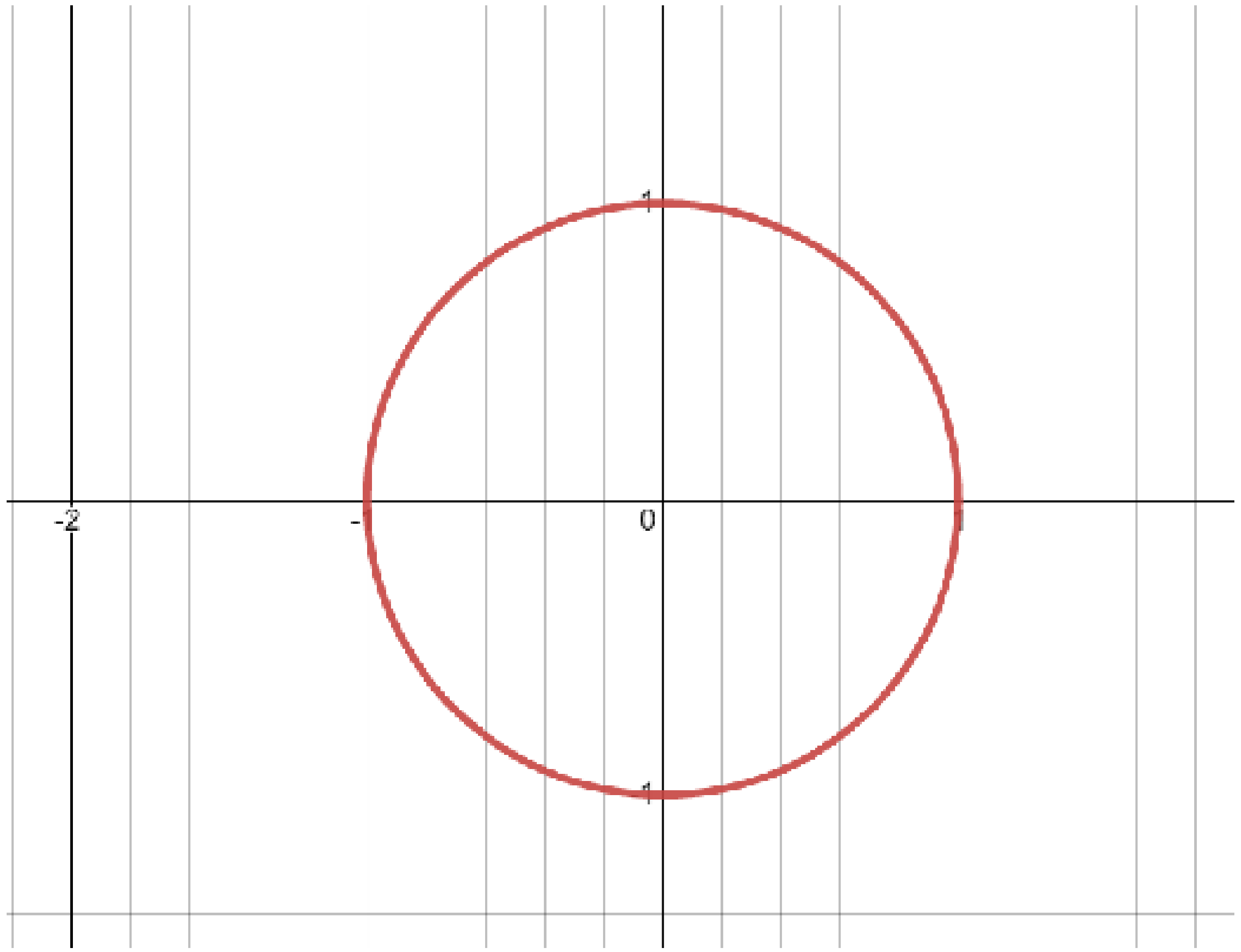
$y = ((-2) + 1)$

$= -1$

EXAMPLE 2 What curve is represented by the following parametric equations?

$$x = \cos t \qquad y = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

t=0	x= 1	y= 0
t=pi/2	x= 0	y= 1
t=pi	x= -1	y=0
t=3pi/2	x=0	y=-1
t=2pi	x=1	y=0



EXAMPLE 3 What curve is represented by the given parametric equations?

$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leqslant t \leqslant 2\pi$$

t	x	y
0	0	1
$\pi/2$	0	-1
π	0	1
$3\pi/2$	0	-1
2π	0	1

$\sin 2(0) = 0$

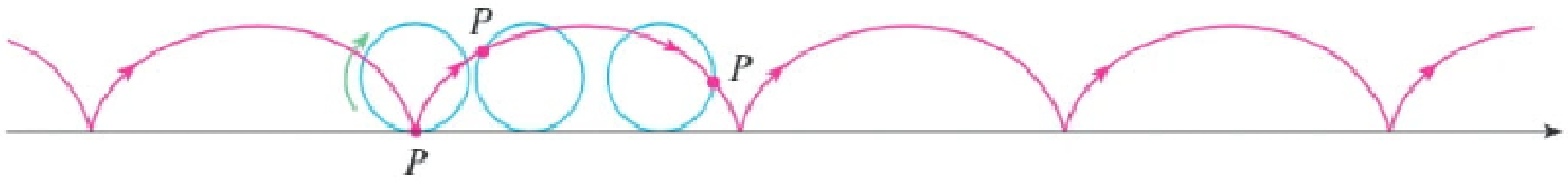
$\sin 2(\pi/2) =$

$\sin \pi = 0$

$\cos \pi = 1$

The Cycloid

EXAMPLE 7 The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a **cycloid** (see Figure 13). If the circle has radius r and rolls along the x-axis and if one position of P is the origin, find parametric equations for the cycloid.



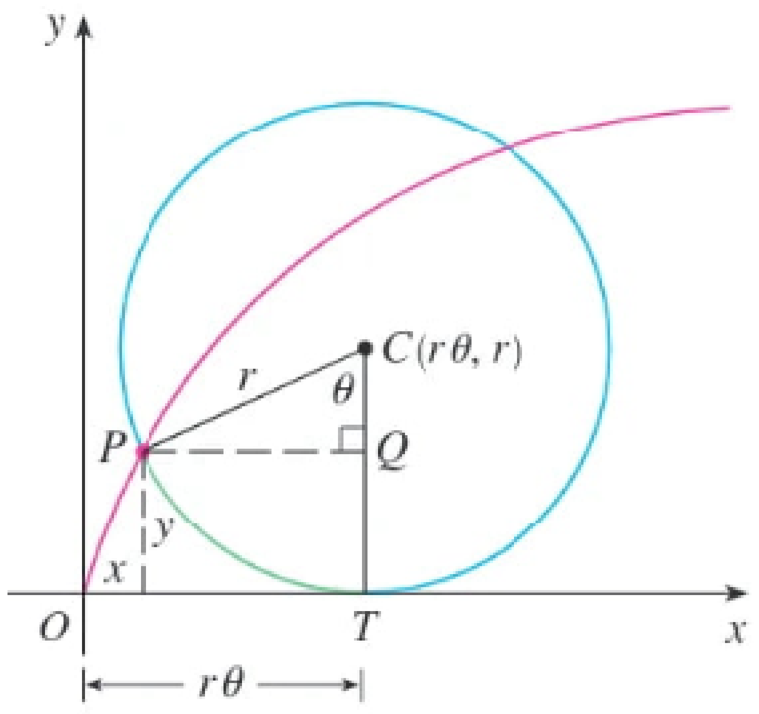


FIGURE 14

SOLUTION We choose as parameter the angle of rotation θ of the circle ($\theta = 0$ when P is at the origin). Suppose the circle has rotated through θ radians. Because the circle has been in contact with the line, we see from Figure 14 that the distance it has rolled from the origin is

$$|OT| = \text{arc } PT = r\theta$$

Therefore the center of the circle is $C(r\theta, r)$. Let the coordinates of P be (x, y). Then from Figure 14 we see that

$$x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$$

$$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$$

Therefore parametric equations of the cycloid are

$$\boxed{1} \qquad x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta) \qquad \theta \in \mathbb{R}$$

Tangents

Suppose f and g are differentiable functions and we want to find the tangent line at a point on the parametric curve $x = f(t)$, $y = g(t)$, where y is also a differentiable function of x. Then the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If $dx/dt \neq 0$, we can solve for dy/dx:

1 $$\boxed{\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} \qquad \text{if} \quad \frac{dx}{dt} \neq 0}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}$$

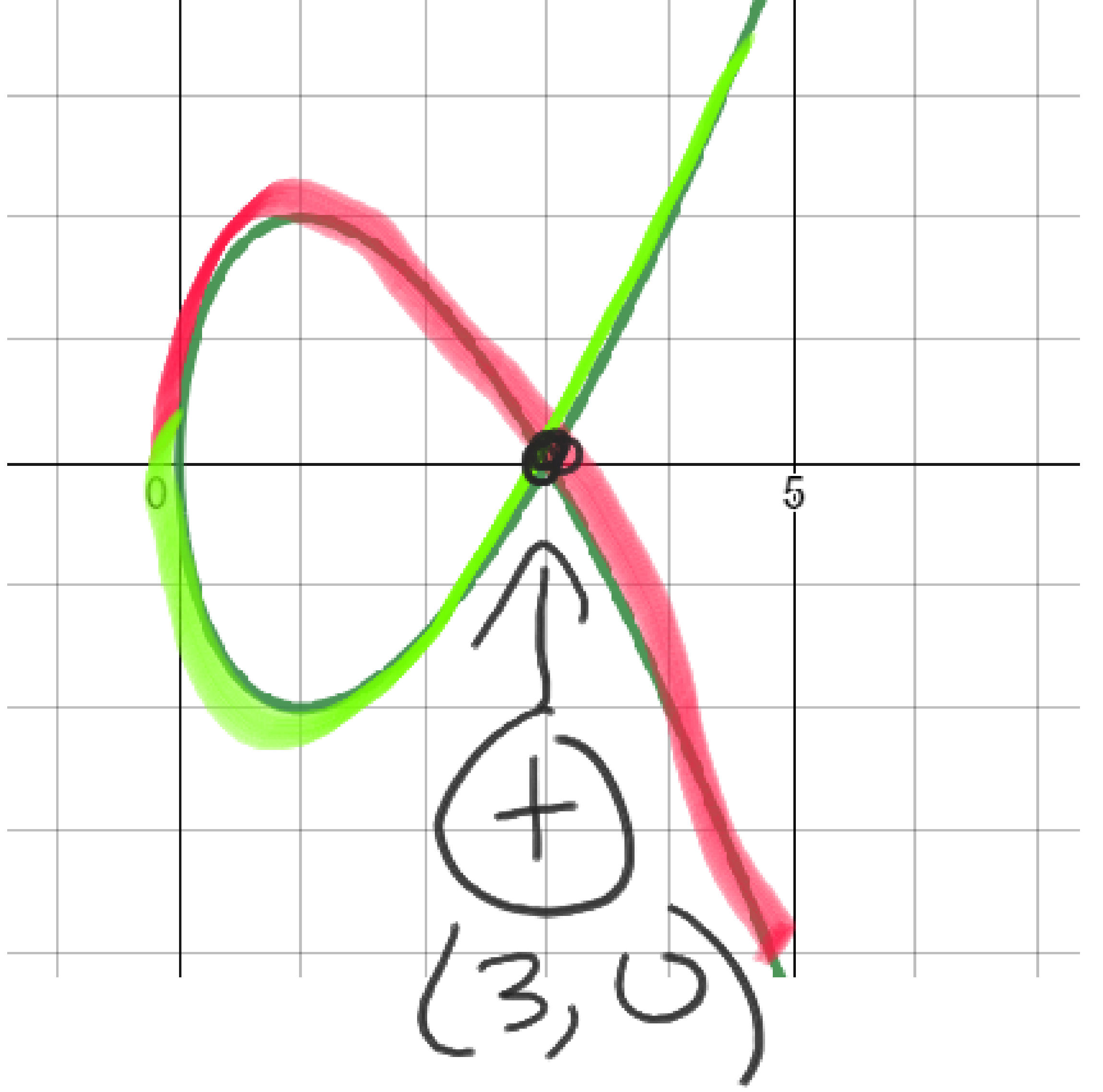
EXAMPLE 1 A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.

(a) Show that C has two tangents at the point $(3, 0)$ and find their equations.

(b) Find the points on C where the tangent is horizontal or vertical.

(c) Determine where the curve is concave upward or downward.

(d) Sketch the curve.



$$3 = t^2$$

$$\pm\sqrt{3} = t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t}$$

$$dy/dt = 3t^2 - 3$$

$$dx/dt = 2t$$

Slope at $\sqrt{3}$

$$\frac{3(\sqrt{3})^2 - 3}{2\sqrt{3}} = \frac{6}{2\sqrt{3}} = \frac{6\sqrt{3}}{6} = \sqrt{3}$$

Slope at $-\sqrt{3}$

$$\frac{3(-\sqrt{3})^2 - 3}{2(-\sqrt{3})} = \frac{6}{-2\sqrt{3}} = -\sqrt{3}$$

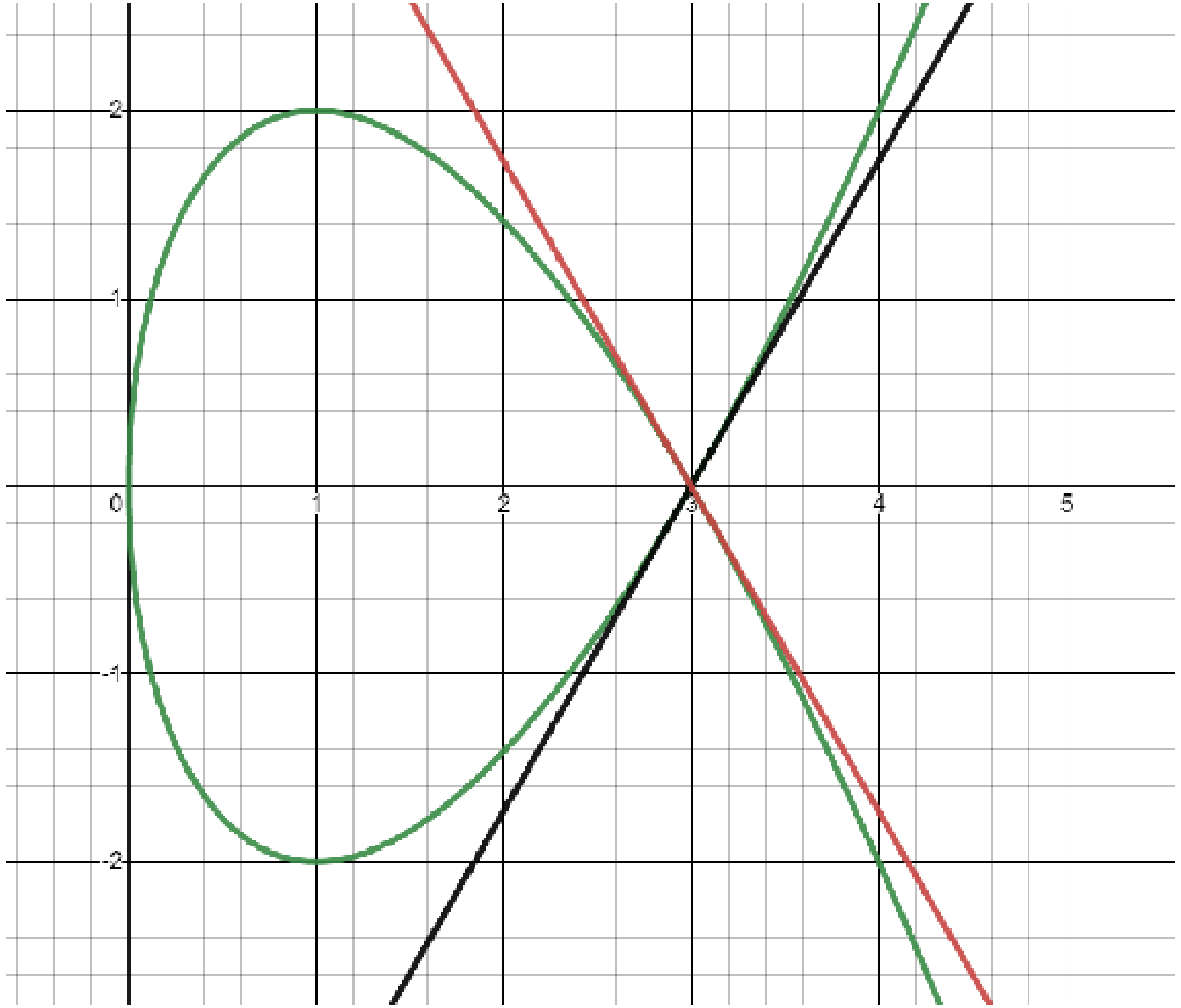
$$(y - y_1) = m(x - x_1)$$

$(3, 0)$ $m = \sqrt{3}, -\sqrt{3}$

$$y - 0 = \sqrt{3}(x - 3)$$

$$y = \sqrt{3}(x - 3)$$

$$y = -\sqrt{3}(x - 3)$$



b) Horizontal = $dy/dt = 0$ / Vertical = $dx/dt = 0$

$$\frac{dy}{dy} = \frac{3t^2 - 3}{2t} \qquad x = t^2 \qquad y = t^3 - 3t$$

Horz → $3t^2 - 3 = 0$

$3t^2 = 3$

$t^2 = 1$

$t = \pm 1$

$t = 1$ $x = 1$ $y = -2$

$(1, -2)$

$t = -1$ $x = 1$ $y = 2$

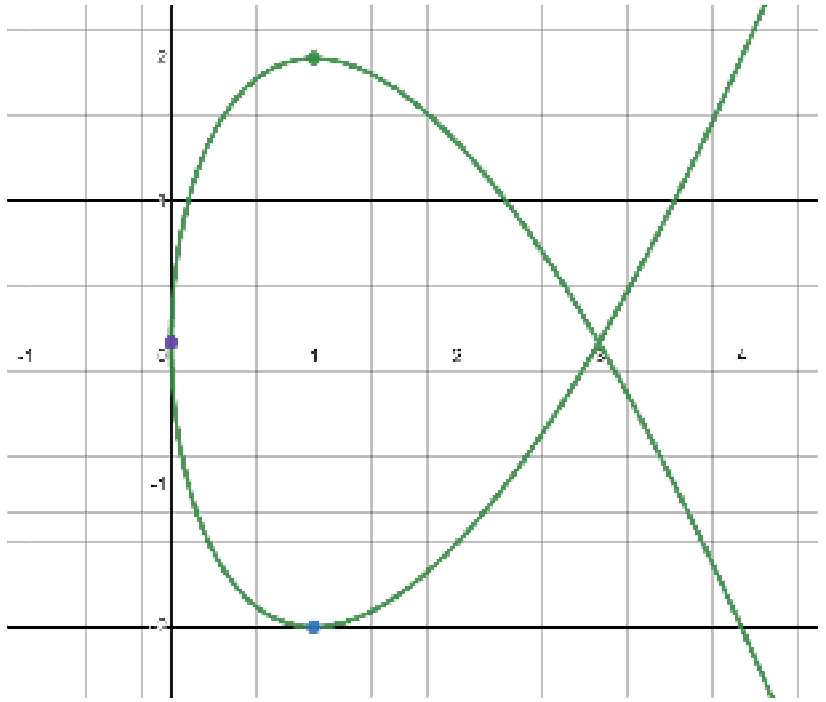
$(1, 2)$

Vert → $2t = 0$

$t = 0$

$t = 0$ $x = 0$ $y = 0$

$(0, 0)$



$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$x = t^2$

$y = t^3 - 3t$

$\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$

$\frac{dy}{dx} = \frac{3t^2}{2t} - \frac{3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$

$t > 0$ concave up

$t < 0$ concave down

$$\frac{\frac{d}{dy}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{3}{2}\left(1 + \frac{1}{t^2}\right)}{2t} = \frac{3\left(\frac{t^2+1}{t^2}\right)}{4t} = \boxed{\frac{3(t^2+1)}{4t^3}}$$

$3(t^2+1) = 0$

$t^2 + 1 = 0$

$-1 \quad -1$

$\sqrt{t^2} = \sqrt{-1}$

$4t^3 = 0$

$t = 0$

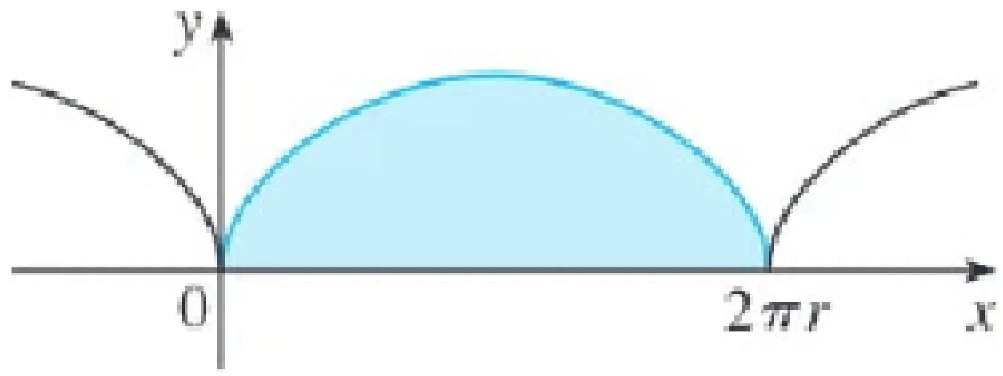
Areas

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)\,dx$, where $F(x) \geqslant 0$. If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt \qquad \left[\text{or} \quad \int_\beta^\alpha g(t)f'(t)\,dt\right]$$

EXAMPLE 3 Find the area under one arch of the cycloid

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta)$$



$\int y\,dx$

$dx = r(1-\cos\theta)$

$$\int_0^{2\pi} (r(1-\cos\theta))(r(1-\cos\theta))\,d\theta$$

$\frac{1}{2}(1+\cos 2\theta)$

$$r^2\int_0^{2\pi}(1-\cos\theta)^2\,d\theta = r^2\int_0^{2\pi}\left(1-2\cos\theta \boxed{+\cos^2\theta}\right)d\theta$$

$$r^2\left[\theta - 2\sin\theta + \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi}$$

$$r^2\left[\theta - 2\sin\theta + \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi}$$

$$r^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

$$r^2(3\pi) = \boxed{3\pi r^2}$$

Arc Length

We already know how to find the length L of a curve C given in the form $y = F(x)$, $a \leqslant x \leqslant b$. Formula 8.1.3 says that if F' is continuous, then

$$\boxed{2} \qquad L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Suppose that C can also be described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where $dx/dt = f'(t) > 0$. This means that C is traversed once, from left to right, as t increases from α to β and $f(\alpha) = a$, $f(\beta) = b$. Putting Formula 1 into Formula 2 and using the Substitution Rule, we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt$$

Since $dx/dt > 0$, we have

$$\boxed{3} \qquad L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

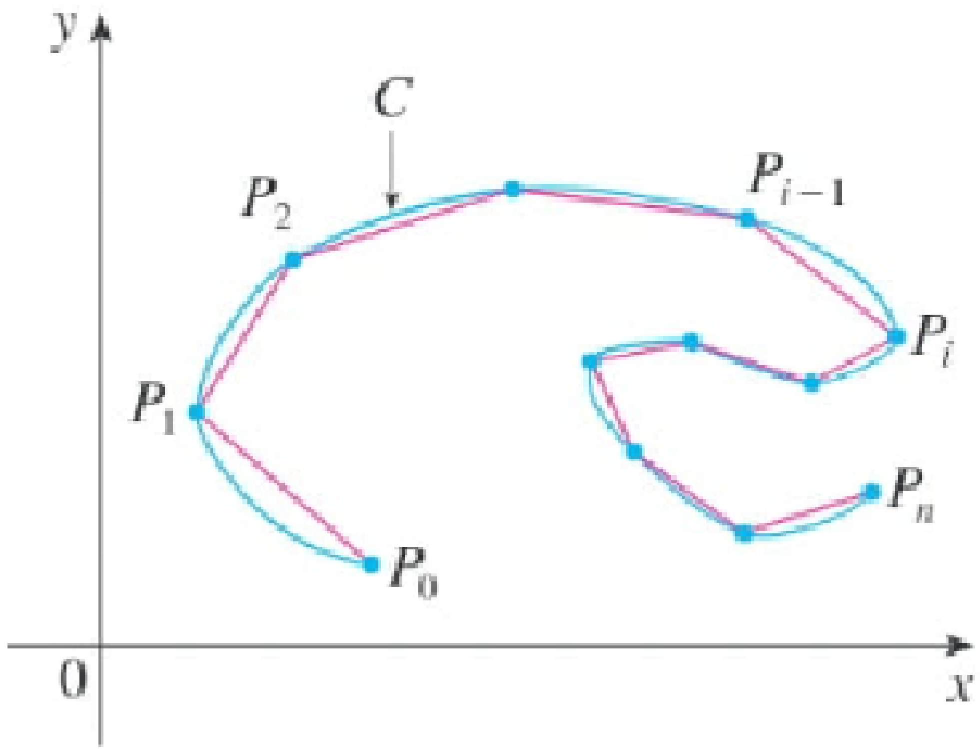


FIGURE 4

EXAMPLE 4 If we use the representation of the unit circle given in Example 10.1.2.

$$x = \cos t \qquad y = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

$$\int_0^{2\pi} \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\, dt$$

$$\int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$\int_0^{2\pi} \sqrt{1}\, dt$$

$$t\Big]_0^{2\pi} = 2\pi$$

$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leqslant t \leqslant 2\pi$$

$$dy/dt = 2\cos 2t$$

$$dy/dt = -2\sin 2t$$

$$\int_0^{2\pi} \sqrt{4\cos^2 2t + 4\sin^2 2t}\, dt$$

$$\int_0^{2\pi} \sqrt{4} = \int_0^{2\pi} 2$$

$$2\Big]_0^{2\pi} = 4\pi$$

Surface Area

In the same way as for arc length, we can adapt Formula 8.2.5 to obtain a formula for surface area. Suppose the curve c given by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where f', g' are continuous, $g(t) \geqslant 0$, is rotated about the x-axis. If C is traversed exactly once as t increases from α to β, then the area of the resulting surface is given by

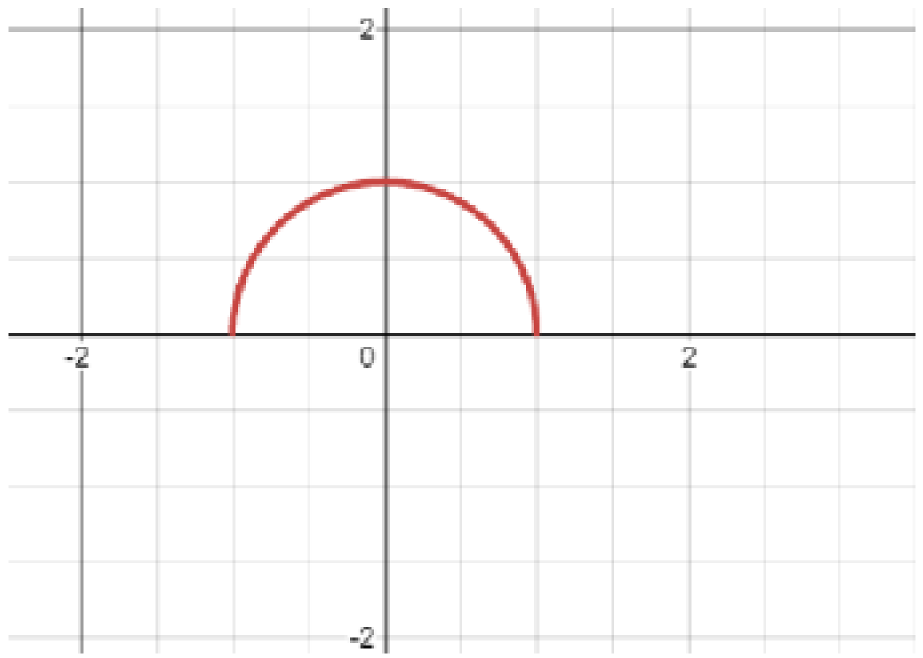
6
$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The general symbolic formulas $S = \int 2\pi y\, ds$ and $S = \int 2\pi x\, ds$ (Formulas 8.2.7 and 8.2.8) are still valid, but for parametric curves we use

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

EXAMPLE 6 Show that the surface area of a sphere of radius r is $4\pi r^2$.

$x = r\cos t \quad y = r\sin t \quad 0 \le t \le \pi$



$$\int_0^{\pi} 2\pi (r\sin t)\sqrt{(-r\sin t)^2 + (r\cos t)^2}\,dt$$

$$2\pi \int_0^{\pi} r\sin t \sqrt{r^2(\sin^2 t + \cos^2 t)}\,dt$$

$$2\pi \int_0^{\pi} r^2 \sin t\,dt = 2\pi r^2 \int_0^{\pi} \sin t\,dt$$

$$2\pi r^2 \left[-\cos t\right]_0^{\pi}$$

$$2\pi r^2 (1+1) = 4\pi r^2$$