$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$$

The Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\sum a_n$ is also divergent.

$$\frac{1}{2^n + 1} < \frac{1}{2^n}$$

$$\frac{1}{2^n} \rightarrow \text{geometric}$$

$$\frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$$

converge

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{2^n + 1}} < 1$$

converges

$$\frac{a}{1-r} = \frac{1/2}{1-1/2} = 1 \qquad r = 1/2 < 1$$

EXAMPLE 1 Determine whether the series $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ converges or diverges.

$$\frac{5}{2n^2} > \frac{5}{2n^2+4n+3}$$

$$\boxed{b_n > a_n}$$

$$\frac{5}{2n^2} = \frac{5}{2}\left(\frac{1}{n^2}\right)$$

$\boxed{\text{p-series } p = 2 > 1 \text{ converges}}$

a_n also converges by comparison test

EXAMPLE 2 Test the series $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ for convergence or divergence.

$$\frac{\ln k}{k} > \frac{1}{k}$$

$\frac{1}{k} \to$ harmonic series diverges

$$a_n > b_n$$

$\uparrow$ diverges

diverges

$$\boxed{\sum_{k=1}^{\infty} \frac{\ln k}{k} \to \text{diverges}}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1} > \frac{1}{2^n}$$

converges

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1} < \frac{1}{2^n}$$

$$a_n < b_n$$

The Limit Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

EXAMPLE 3 Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

$$a_n = \frac{1}{2^n - 1} \qquad b_n = \frac{1}{2^n}$$

$$\lim_{n \to \infty} \frac{1/2^n - 1}{1/2^n} = \frac{1}{2^n - 1} \cdot \frac{2^n}{1} = \frac{2^n/2^n}{2^n - 1/2^n} = \frac{1}{1 - 1/2^n}$$

$$\lim_{n \to \infty} \frac{1}{1 - 1/2^n} = 1 > 0$$

$$\boxed{\text{Since } b_n \text{ converges} \quad a_n \text{ converges}}$$

EXAMPLE 4 Determine whether the series $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ converges or diverges.

$$a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}$$

$$b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$$

p-series $p = 1/2 < 1$

diverges

$$\lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} \cdot \frac{n^{1/2}}{2} = \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{5+n^5}} \frac{/n^{5/2}}{/n^{5/2}}$$

$$\lim_{n\to\infty} \frac{2+3/n}{2\sqrt{5/n^5+1}} \rightarrow \frac{2}{2} = 1 > 0$$

a_n diverges

$$\frac{\sqrt{5+n^5}}{n^{5/2}} = \frac{\sqrt{5+n^5}}{\sqrt{n^5}} = \sqrt{\frac{5+n^5}{n^5}}$$

$$= \sqrt{5/n^5 + 1}$$

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n+1}$$

We see from these examples that the nth term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n \qquad \text{or} \qquad a_n = (-1)^n b_n$$

where b_n is a positive number. (In fact, $b_n = |a_n|$.)

$$a_n = \boxed{(-1)^{n-1}} b_n$$

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies $\sum_{n=1} (-1)^n b_n =$

(i) $b_{n+1} \leqslant b_n$ for all n

(ii) $\lim_{n \to \infty} b_n = 0$

then the series is convergent.

$\boxed{7}$ **Test for Divergence** If $\lim_{n \to \infty} a_n$ does not exist or if $\lim_{n \to \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

EXAMPLE 1 The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

i $\quad b_{n+1} \le b_n$

$$\frac{1}{n+1} \le \frac{1}{n} \quad ✓$$

ii $\quad \lim_{n \to \infty} \frac{1}{n} = 0 \quad ✓$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

harmonic series

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges

alternating harmonic series

EXAMPLE 2 The series $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$

alternating $(-1)^n$

$$b_n = \frac{3n}{4n-1} > 0$$

i $b_{n+1} \leq b_n$

ii $\lim_{n \to \infty} b_n = 0$

ii) $\lim_{n \to \infty} \frac{3n / n}{4n-1 / n}$

$$\lim_{n \to \infty} \frac{3}{4 - 1/n} = \frac{3}{4} \neq 0$$

fails Alt Test

passes divergence test

EXAMPLE 3 Test the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ for convergence or divergence.

✓

$$b_n = \frac{n^2}{n^3+1}$$

$$\lim_{n \to \infty} \frac{n^2 / n^3}{n^3+1 / n^3} = \frac{1/n}{1 + 1/n^3} = \frac{0}{1} = 0$$ ii passed

$$f'(x) = \frac{2x(x^3+1) - 3x^2(x^2)}{(x^3+1)^2} < 0$$ i passed

$$= 2x^4 + 2x - 3x^4 = 2x - x^4$$

$$= x(2 - x^3) < 0$$

$x < 0$

$x > \sqrt[3]{2}$

$$x(2-x^3) < 0$$

$$x(2-x^3) = 0$$

$$x = 0$$

$$2 - x^3 = 0$$

$$-x^3 = -2$$

$$x^3 = 2$$

$$x = \sqrt[3]{2}$$

$$\cancel{x < 0} \qquad \boxed{x > \sqrt[3]{2}}$$