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Infinite Sequences and Series



11.6

Absolute Convergence and the Ratio and Root Tests

Absolute Convergence and the Ratio and Root Tests

Given any series $\sum a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute values of the terms of the original series.

1 Definition A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Absolute Convergence and the Ratio and Root Tests

Notice that if $\sum a_n$ is a series with positive terms, then $|a_n| = a_n$ and so absolute convergence is the same as convergence in this case.

Example 1

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

is a convergent p -series ($p = 2$).

Example 2

We know that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

is convergent, but it is not absolutely convergent because the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

which is the harmonic series (p -series with $p = 1$) and is therefore divergent.

Absolute Convergence and the Ratio and Root Tests

2 Definition A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Example 2 shows that the alternating harmonic series is conditionally convergent. Thus it is possible for a series to be convergent but not absolutely convergent.

However, the next theorem shows that absolute convergence implies convergence.

3 Theorem If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Example 3

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \dots$$

is convergent or divergent.

Solution:

This series has both positive and negative terms, but it is not alternating. (The first term is positive, the next three are negative, and the following three are positive: The signs change irregularly.)

Example 3 – Solution

cont'd

We can apply the Comparison Test to the series of absolute values

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

Since $|\cos n| \leq 1$ for all n , we have

$$\frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$$

We know that $\sum 1/n^2$ is convergent (p -series with $p = 2$) and therefore $\sum |\cos n|/n^2$ is convergent by the Comparison Test.

Example 3 – *Solution*

cont'd

Thus the given series $\sum (\cos n)/n^2$ is absolutely convergent and therefore convergent by Theorem 3.

Absolute Convergence and the Ratio and Root Tests

The following test is very useful in determining whether a given series is absolutely convergent.

The Ratio Test

(i) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Absolute Convergence and the Ratio and Root Tests

Note:

Part (iii) of the Ratio Test says that if $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = 1$, the test gives no information. For instance, for the convergent series $\sum 1/n^2$ we have

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \frac{1}{\left(1 + \frac{1}{n}\right)^2} \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

Absolute Convergence and the Ratio and Root Tests

whereas for the divergent series $\sum 1/n$ we have

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}} \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

Therefore, if $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = 1$, the series $\sum a_n$ might converge or it might diverge. In this case the Ratio Test fails and we must use some other test.

Example 5

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

Solution:

Since the terms $a_n = n^n/n!$ are positive, we don't need the absolute value signs.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \\ &= \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n}\end{aligned}$$

Example 5 – *Solution*

cont'd

$$= \left(\frac{n+1}{n} \right)^n = \left(1 + \frac{1}{n} \right)^n \rightarrow e \quad \text{as } n \rightarrow \infty$$

Since $e > 1$, the given series is divergent by the Ratio Test.

Absolute Convergence and the Ratio and Root Tests

Note:

Although the Ratio Test works in Example 5, an easier method is to use the Test for Divergence. Since

$$a_n = \frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n} \geq n$$

it follows that a_n does not approach 0 as $n \rightarrow \infty$. Therefore the given series is divergent by the Test for Divergence.

Absolute Convergence and the Ratio and Root Tests

The following test is convenient to apply when n th powers occur.

The Root Test

- (i) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
- (ii) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- (iii) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

Absolute Convergence and the Ratio and Root Tests

If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$, then part (iii) of the Root Test says that the test gives no information. The series $\sum a_n$ could converge or diverge.

(If $L = 1$ in the Ratio Test, don't try the Root Test because L will again be 1. And if $L = 1$ in the Root Test, don't try the Ratio Test because it will fail too.)

Example 6

Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2} \right)^n$.

Solution:

$$a_n = \left(\frac{2n+3}{3n+2} \right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2 + \frac{3}{n}}{3 + \frac{2}{n}} \rightarrow \frac{2}{3} < 1$$

Thus the given series is absolutely convergent (and therefore convergent) by the Root Test.



Rearrangements

Rearrangements

The question of whether a given convergent series is absolutely convergent or conditionally convergent has a bearing on the question of whether infinite sums behave like finite sums.

If we rearrange the order of the terms in a finite sum, then of course the value of the sum remains unchanged. But this is not always the case for an infinite series.

Rearrangements

By a **rearrangement** of an infinite series $\sum a_n$ we mean a series obtained by simply changing the order of the terms. For instance, a rearrangement of $\sum a_n$ could start as follows:

$$a_1 + a_2 + a_5 + a_3 + a_4 + a_{15} + a_6 + a_7 + a_{20} + \cdots$$

It turns out that

if $\sum a_n$ is absolutely convergent series with sum s , then any rearrangement of $\sum a_n$ has the same sum s .

Rearrangements

However, any conditionally convergent series can be rearranged to give a different sum. To illustrate this fact let's consider the alternating harmonic series

$$\boxed{6} \quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \ln 2$$

If we multiply this series by $\frac{1}{2}$, we get

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots = \frac{1}{2} \ln 2$$

Rearrangements

Inserting zeros between the terms of this series, we have

$$\boxed{7} \quad 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \cdots = \frac{1}{2} \ln 2$$

Now we add the series in Equations 6 and 7:

$$\boxed{8} \quad 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3}{2} \ln 2$$

Rearrangements

Notice that the series in (8) contains the same terms as in (6) but rearranged so that one negative term occurs after each pair of positive terms. The sums of these series, however, are different. In fact, Riemann proved that

if $\sum a_n$ is a conditionally convergent series and r is any real number whatsoever, then there is a rearrangement of $\sum a_n$ that has a sum equal to r .