Chapter 11 Practice Test

Test the series for convergence or divergence

1)

$$\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$$

$\frac{n-1}{n^3+1} < \frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}$ for $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges because it is a p-series with $p = 2 > 1$.

2)

$$\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2 n}$$

$0 \leq n\cos^2 n \leq n$, so $\frac{1}{n + n\cos^2 n} \geq \frac{1}{n+n} = \frac{1}{2n}$. Thus, $\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2 n}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{2n}$, which is a constant multiple of the (divergent) harmonic series.

3)

$$\sum_{n=1}^{\infty} \frac{\sin 2n}{1 + 2^n}$$

$\left|\frac{\sin 2n}{1+2^n}\right| \leq \frac{1}{1+2^n} < \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$, so the series $\sum_{n=1}^{\infty} \left|\frac{\sin 2n}{1+2^n}\right|$ converges by comparison with the geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ with $|r| = \frac{1}{2} < 1$. Thus, the series $\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$ converges absolutely, implying convergence.

4)

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2 - 1}{n^2 + 1}$$

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left|(-1)^n \frac{n^2-1}{n^2+1}\right| = \lim_{n\to\infty} \frac{1-1/n^2}{1+1/n^2} = 1 \neq 0$, so the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1}$ diverges by the Test for Divergence. $\left[\text{Note that } \lim_{n\to\infty} (-1)^n \frac{n^2-1}{n^2+1} \text{ does not exist.}\right]$

5)

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^4}{4^n}$$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^4}{4^{n+1}} \cdot \frac{4^n}{n^4}\right| = \lim_{n\to\infty} \frac{(n+1)^4}{4n^4} = \frac{1}{4} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^4 = \frac{1}{4}(1) = \frac{1}{4} < 1$, so the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^4}{4^n}$ is absolutely convergent (and therefore convergent) by the Ratio Test.

6)

$$\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$$

Use the Ratio Test. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!}\right| = \lim_{n\to\infty} \frac{(n+1)n! \cdot e^{n^2}}{e^{n^2+2n+1} n!} = \lim_{n\to\infty} \frac{n+1}{e^{2n+1}} = 0 < 1$, so $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$ converges.

7)

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2}$$

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n^2/n} = \lim_{n\to\infty} \frac{1}{[(n+1)/n]^n} = \frac{1}{\lim_{n\to\infty}(1+1/n)^n} = \frac{1}{e} < 1$, so the series $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ converges by the Root Test.

8)

$$\sum_{n=1}^{\infty} n^2 e^{-n^3}$$

Let $f(x) = x^2 e^{-x^3}$. Then f is continuous and positive on $[1, \infty)$, and $f'(x) = \frac{x(2-3x^3)}{e^{x^3}} < 0$ for $x \geq 1$, so f is decreasing on $[1, \infty)$ as well, and we can apply the Integral Test. $\int_1^{\infty} x^2 e^{-x^3}\, dx = \lim_{t\to\infty} \left[-\frac{1}{3}e^{-x^3}\right]_1^t = \frac{1}{3e}$, so the integral converges, and hence, the series converges.

9)

$$\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^n$$

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} (2^{1/n} - 1) = 1 - 1 = 0 < 1$, so the series $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^n$ converges by the Root Test.

Find the radius of convergence and interval of convergence of the series

10)

$$\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$$

If $a_n = \dfrac{x^n}{2n-1}$, then $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{2n+1}\cdot\frac{2n-1}{x^n}\right| = \lim_{n\to\infty}\left(\frac{2n-1}{2n+1}|x|\right) = \lim_{n\to\infty}\left(\frac{2-1/n}{2+1/n}|x|\right) = |x|$. By the Ratio Test, the series $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{2n-1}$ converges when $|x|<1$, so $R=1$. When $x=1$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n-1}$ diverges by comparison with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n}$ since $\dfrac{1}{2n-1} > \dfrac{1}{2n}$ and $\displaystyle\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges since it is a constant multiple of the harmonic series. When $x=-1$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}$ converges by the Alternating Series Test. Thus, the interval of convergence is $[-1,1)$.

11)

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$$

If $a_n = \dfrac{(x-2)^n}{n^2+1}$, then $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^2+1}\cdot\frac{n^2+1}{(x-2)^n}\right| = |x-2|\lim_{n\to\infty}\frac{n^2+1}{(n+1)^2+1} = |x-2|$. By the Ratio Test, the series $\displaystyle\sum_{n=0}^{\infty}\frac{(x-2)^n}{n^2+1}$ converges when $|x-2|<1$ $[R=1]$ $\Leftrightarrow$ $-1<x-2<1$ $\Leftrightarrow$ $1<x<3$. When $x=1$, the series $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1}{n^2+1}$ converges by the Alternating Series Test; when $x=3$, the series $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n^2+1}$ converges by comparison with the p-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ $[p=2>1]$. Thus, the interval of convergence is $I=[1,3]$.

Find a power series representation for the function and determine the radius of convergence.

12)

$$f(x) = \frac{x}{(1 + 4x)^2}$$

We know that $\frac{1}{1+4x} = \frac{1}{1-(-4x)} = \sum_{n=0}^{\infty} (-4x)^n$. Differentiating, we get

$$\frac{-4}{(1+4x)^2} = \sum_{n=1}^{\infty} (-4)^n n x^{n-1} = \sum_{n=0}^{\infty} (-4)^{n+1}(n+1)x^n, \text{ so}$$

$$f(x) = \frac{x}{(1+4x)^2} = \frac{-x}{4} \cdot \frac{-4}{(1+4x)^2} = \frac{-x}{4} \sum_{n=0}^{\infty} (-4)^{n+1}(n+1)x^n = \sum_{n=0}^{\infty} (-1)^n 4^n (n+1) x^{n+1}$$

for $|-4x| < 1 \quad \Leftrightarrow \quad |x| < \frac{1}{4}$, so $R = \frac{1}{4}$.

13)

$$f(x) = \frac{x^2}{x^4 + 16}$$

$$f(x) = \frac{x^2}{x^4+16} = \frac{x^2}{16}\left(\frac{1}{1+x^4/16}\right) = \frac{x^2}{16}\left(\frac{1}{1-[-(x/2)]^4}\right) = \frac{x^2}{16} \sum_{n=0}^{\infty} \left[-\left(\frac{x}{2}\right)^4\right]^n \text{ or, equivalently, } \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2^{4n+4}}.$$

The series converges when $\left|-\left(\frac{x}{2}\right)^4\right| < 1 \quad \Rightarrow \quad \left|\frac{x}{2}\right| < 1 \quad \Rightarrow \quad |x| < 2$, so $R = 2$ and $I = (-2, 2)$.

Use a Maclaurin series in Table 1 to obtain the Maclaurin series for the given function.

14)

$$f(x) = \sin(\pi x/4)$$

$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, so $f(x) = \sin\left(\frac{\pi}{4}x\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{4}x\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{4^{2n+1}(2n+1)!} x^{2n+1}$, $R = \infty$.