Chapter 11

Infinite Sequences and Series

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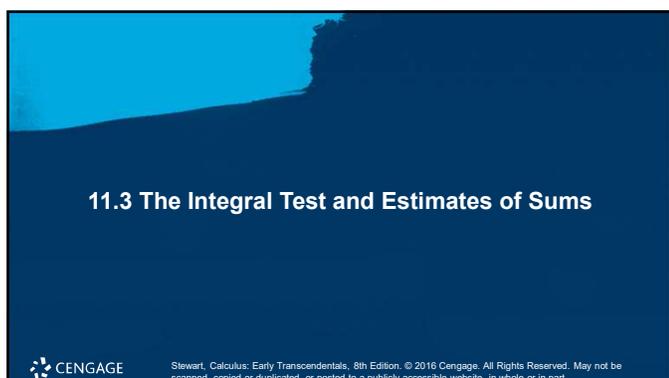
11.3 The Integral Test and Estimates of Sums

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The Integral Test and Estimates of Sums (1 of 11)

In general, it is difficult to find the exact sum of a series.

We were able to accomplish this for geometric series and the series $\sum \frac{1}{n(n+1)}$

because in each of those cases we could find a simple formula for the nth partial sum s_n.

But usually it isn't easy to discover such a formula.

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The Integral Test and Estimates of Sums (2 of 11)

We begin by investigating the series whose terms are the reciprocals of the squares of the positive integers:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

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The Integral Test and Estimates of Sums (3 of 11)

There's no simple formula for the sum s_n of the first n terms, but the computer-generated table of approximate values given in the margin suggests that the partial sums are approaching a number near 1.64 as $n \to \infty$ and so it looks as if the series is convergent.

n	$s_n = \sum_{i=1}^{n}\frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

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The Integral Test and Estimates of Sums (4 of 11)

We can confirm this impression with a geometric argument. Figure 1 shows the curve $y = \frac{1}{x^2}$ and rectangles that lie below the curve.
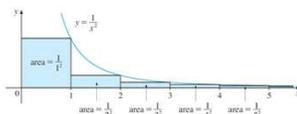

Figure 1

The base of each rectangle is an interval of length 1; the height is equal to the value of the function $y = \frac{1}{x^2}$ at the right endpoint of the interval.

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The Integral Test and Estimates of Sums (5 of 11)

So the sum of the areas of the rectangles is

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

If we exclude the first rectangle, the total area of the remaining rectangles is smaller than the area under the curve $y = \frac{1}{x^2}$ for $x \geq 1$, which is the value of the integral $\int_{1}^{\infty} \left(\frac{1}{x^2} \right) dx$.

This improper integral is convergent and has value 1. So the picture shows that all the partial sums are less than

$$\frac{1}{1^2} + \int_{1}^{\infty} \frac{1}{x^2} dx = 2$$

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The Integral Test and Estimates of Sums (6 of 11)

Thus the partial sums are bounded. We also know that the partial sums are increasing (because all the terms are positive). Therefore the partial sums converge (by the Mono tonic Sequence Theorem) and so the series is convergent. The sum of the series (the limit of the partial sums) is also less than 2:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots < 2$$

Now let's look at the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots$$

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The Integral Test and Estimates of Sums (7 of 11)

The table of values of s_n suggests that the partial sums aren't approaching a finite number, so we suspect that the given series may be divergent.

n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

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The Integral Test and Estimates of Sums (8 of 11)

Again we use a picture for confirmation. Figure 2 shows the curve $y = \frac{1}{\sqrt{x}}$, but this time we use rectangles whose tops lie *above* the curve.
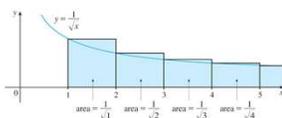

Figure 2

The base of each rectangle is an interval of length 1. The height is equal to the value of the function $y = \frac{1}{\sqrt{x}}$ at the *left* endpoint of the interval.

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The Integral Test and Estimates of Sums (9 of 11)

So the sum of the areas of all the rectangles is

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

This total area is greater than the area under the curve $y = \frac{1}{\sqrt{x}}$, for $x \geq 1$, which is equal to the integral $\int_{1}^{\infty} \left(\frac{1}{\sqrt{x}} \right) dx$.

But we know that this improper integral is divergent. In other words, the area under the curve is infinite. So the sum of the series must be infinite, that is, the series is divergent.

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The Integral Test and Estimates of Sums (10 of 11)

The same sort of geometric reasoning that we used for these two series can be used to prove the following test.

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_{1}^{\infty} f(x)\, dx$ is convergent. In other words:

(i) If $\int_{1}^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_{1}^{\infty} f(x)\, dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

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Example 1

Test the series $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ for convergence or divergence.

Solution:

The function $f(x)=\frac{1}{(x^2+1)}$ is continuous, positive, and decreasing on $[1,\infty)$

so we use the Integral Test:

$$\int_1^{\infty}\frac{1}{x^2+1}\,dx = \lim_{t\to\infty}\int_1^{t}\frac{1}{x^2+1}\,dx = \lim_{t\to\infty}\tan^{-1}x\,\Big]_1^{t}$$

$$= \lim_{t\to\infty}\left(\tan^{-1}t-\frac{\pi}{2}\right)=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

Thus $\int_1^{\infty}\frac{1}{(x^2+1)}\,dx$ is a convergent integral and so, by the Integral Test, the series $\sum\frac{1}{(x^2+1)}$ is convergent.

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The Integral Test and Estimates of Sums (11 of 11)

The series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is called the **p-series**.

1 The p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$.

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Example 3

(a) The series

$$\sum_{n=1}^{\infty}\frac{1}{n^3}=\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\cdots$$

is convergent because it is a p-series with $p = 3 > 1$.

(b) The series

$$\sum_{n=1}^{\infty}\frac{1}{n^{1/3}}=\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}}=1+\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{3}}+\frac{1}{\sqrt[3]{4}}+\cdots$$

is divergent because it is a p-series with $p=\frac{1}{3}<1$.

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Example 4 (1 of 2)

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Solution:

The function $f(x) = \frac{(\ln x)}{x}$ is positive and continuous for $x > 1$ because the logarithm function is continuous.

But it is not obvious whether or not f is decreasing, so we compute its derivative:

$$f'(x) = \frac{x\left(\frac{1}{x}\right) - \ln x}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

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Example 4 (2 of 2)

Thus $f'(x) < 0$ when $\ln x > 1$, that is, $x > e$. It follows that f is decreasing when $x > e$ and so we can apply the Integral Test:

$$\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{t \to \infty} \int_1^{t} \frac{\ln x}{x}\,dx = \lim_{t \to \infty} \frac{(\ln x)^2}{2} \Big|_1^{t}$$

$$= \lim_{t \to \infty} \frac{(\ln t)^2}{2}$$

$$= \infty$$

Since this improper integral is divergent, the series $\sum \frac{(\ln n)}{n}$ is also divergent by the Integral Test.

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11.4 The Comparison Tests

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The Comparison Tests (1 of 8)

In the comparison tests the idea is to compare a given series with a series that is known to be convergent or divergent. For instance, the series

$$\textbf{1} \quad \sum_{n=1}^{\infty} \frac{1}{2^n + 1}$$

reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$, which is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$ and is therefore convergent. Because the series (1) is so similar to a convergent series, we have the feeling that it too must be convergent. Indeed, it is.

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The Comparison Tests (2 of 8)

The inequality

$$\frac{1}{2^n + 1} < \frac{1}{2^n}$$

shows that our given series (1) has smaller terms than those of the geometric series and therefore all its partial sums are also smaller than 1 (the sum of the geometric series).

This means that its partial sums form a bounded increasing sequence, which is convergent. It also follows that the sum of the series is less than the sum of the geometric series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1} < 1$$

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The Comparison Tests (3 of 8)

Similar reasoning can be used to prove the following test, which applies only to series whose terms are positive. The first part says that if we have a series whose terms are *smaller* than those of a known *convergent* series, then our series is also convergent.

The second part says that if we start with a series whose terms are *larger* than those of a known *divergent* series, then it too is divergent.

The Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

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The Comparison Tests (4 of 8)

In using the Comparison Test we must, of course, have some known series $\sum b_n$ for the purpose of comparison. Most of the time we use one of these series:

- A p-series $[\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1]$
- A geometric series $\left[\sum ar^{n-1}$ converges if $|r| < 1$ and diverges $|r| \geq 1\right]$

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Example 1

Determine whether the series $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ converges or diverges.

Solution:

For large n the dominant term in the denominator is $2n^2$, so we compare the given series with the series $\sum \frac{5}{(2n^2)}$.

Observe that

$$\frac{5}{2n^2 + 4n + 3} < \frac{5}{2n^2}$$

because the left side has a bigger denominator.

(In the notation of the Comparison Test, a_n is the left side and b_n is the right side.)

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Example 1 – *Solution*

We know that

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent because it's a constant times a p-series with $p = 2 > 1$.

Therefore

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$$

is convergent by part (i) of the Comparison Test.

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The Comparison Tests (5 of 8)

Note 1:

Although the condition $a_n \le b_n$ or $a_n \ge b_n$ in the Comparison Test is given for all n, we need verify only that it holds for $n \ge N$, where N is some fixed integer, because the convergence of a series is not affected by a finite number of terms.

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The Comparison Tests (6 of 8)

Note 2:

The terms of the series being tested must be smaller than those of a convergent series or larger than those of a divergent series.

If the terms are larger than the terms of a convergent series or smaller than those of a divergent series, then the Comparison Test doesn't apply.

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The Comparison Tests (7 of 8)

Consider, for instance, the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

The inequality

$$\frac{1}{2^n - 1} > \frac{1}{2^n}$$

is useless as far as the Comparison Test is concerned because $\sum b_n = \sum \left(\frac{1}{2}\right)^n$ is convergent and $a_n > b_n$.

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The Comparison Tests (8 of 8)

Nonetheless, we have the feeling that $\sum \dfrac{1}{(2^n-1)}$ ought to be convergent because it is very similar to the convergent geometric series $\sum \left(\dfrac{1}{2}\right)^n$.

In such cases the following test can be used.

The Limit Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

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Example 3

Test the series $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{2^n - 1}$ for convergence or divergence.

Solution:

We use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1} \quad b_n = \frac{1}{2^n}$$

and obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{1}{(2^n-1)}}{\dfrac{1}{2^n}}$$

$$= \lim_{n\to\infty} \frac{2^n}{2^n - 1}$$

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Example 3 – *Solution*

$$= \lim_{n\to\infty} \frac{1}{1 - \dfrac{1}{2^n}}$$

$$= 1 > 0$$

Since this limit exists and $\sum \dfrac{1}{2^n}$ is a convergent geometric series, the given series converges by the Limit Comparison Test.

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