Chapter 11
Infinite Sequences and Series

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11.7 Strategy for Testing Series

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Strategy for Testing Series (1 of 5)

We now have several ways of testing a series for convergence or divergence; the problem is to decide which test to use on which series. In this respect, testing series is similar to integrating functions.

Again there are no hard and fast rules about which test to apply to a given series, but you may find the following advice of some use.

It is not wise to apply a list of the tests in a specific order until one finally works. That would be a waste of time and effort. Instead, as with integration, the main strategy is to classify the series according to its *form*.

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Strategy for Testing Series (2 of 5)

1. If the series is of the form $\sum \frac{1}{n^p}$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leq 1$.
2. If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geq 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.

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Strategy for Testing Series (3 of 5)

3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series.

   The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, then we can apply the Comparison Test to $\sum |a_n|$ and test for absolute convergence.

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Strategy for Testing Series (4 of 5)

4. If you can see at a glance that $\lim_{n \to \infty} a_n \neq 0$, then the Test for Divergence should be used.
5. If the series is of the form $\sum (-1)^{n-1} b_n$ or $\sum (-1)^n b_n$, then the Alternating Series Test is an obvious possibility.
6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $\left| \frac{a_{n+1}}{a_n} \right| \to 1$ as $n \to \infty$ for all p-series and therefore all rational algebraic functions of n. Thus the Ratio Test should not be used for such series.

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Strategy for Testing Series (5 of 5)

7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.
8. If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

In the following examples we don't work out all the details but simply indicate which tests should be used.

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Example 1

$$\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$$

Since as $a_n \to \frac{1}{2} \neq 0$ as $n \to \infty$, we should use the Test for Divergence.

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Example 2

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$$

Since a_n is an algebraic function of n, we compare the given series with a p-series.

The comparison series for the Limit Comparison Test is $\sum b_n$, where

$$b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{n^{\frac{3}{2}}}{3n^3} = \frac{1}{3n^{\frac{3}{2}}}$$

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Example 4

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{n^4+1}$$

Since the series is alternating, we use the Alternating Series Test.

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Example 5

$$\sum_{k=1}^{\infty}\frac{2^k}{k!}$$

Since the series involves $k!$, we use the Ratio Test.

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Example 6

$$\sum_{n=1}^{\infty}\frac{1}{2+3^n}$$

Since the series is closely related to the geometric series $\sum \frac{1}{3^n}$, we use the Comparison Test.

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