Strategy for Testing Series (2 of 5)

1. If the series is of the form $\sum \frac{1}{n^p}$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \le 1$.

2. If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geq 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.

Strategy for Testing Series (3 of 5)

3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series.

   The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, then we can apply the Comparison Test to $\sum |a_n|$ and test for absolute convergence.

Strategy for Testing Series (4 of 5)

4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.

5. If the series is of the form $\sum(-1)^{n-1} b_n$ or $\sum(-1)^n b_n$, then the Alternating Series Test is an obvious possibility.

6. Series that involve factorials or other products (including a constant raised to the *n*th power) are often conveniently tested using the Ratio Test. Bear in mind that $\left|\frac{a_{n+1}}{a_n}\right| \to 1$ as $n \to \infty$ for all *p*-series and therefore all rational algebraic functions of *n*. Thus the Ratio Test should not be used for such series.

Strategy for Testing Series (5 of 5)

7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.
8. If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

In the following examples we don't work out all the details but simply indicate which tests should be used.

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1}$

$$\lim_{n\to\infty} \frac{n^2-1}{n^2+1} \to \frac{1-1/n^2}{1+1/n^2} = 1 \neq 0$$

$$\lim_{n\to\infty} (-1)^n \frac{n^2-1}{n^2+1} = \text{DNE}$$

Diverges

T.o.D

Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

(i) $b_{n+1} \leqslant b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

7 Test for Divergence

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx \to \int_2^{\infty} u^{-1/2}\,du = 2u^{1/2} = 2\sqrt{\ln x}\Big]_2^{\infty}$$

$u = \ln(x)$

$du = \frac{1}{x}$

$$\lim_{t\to\infty}\left(2\sqrt{\ln t} - 2\sqrt{\ln 2}\right) = \infty$$

diverges

The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(i) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

9. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$

The Ratio Test

(i) If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

9. $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$

$2(n+1) = 2n+2$

$$\lim_{n\to\infty} \left| \frac{\pi^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{\pi^{2n}} \right|$$

$$\frac{\cancel{\pi^{2n}} \cdot \pi^2 \; \cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!} \cdot \cancel{\pi^{2n}}} = \frac{\pi^2}{(2n+2)(2n+1)}$$

$$\lim_{n\to\infty} \frac{\pi^2}{(2n+2)(2n+1)} = 0 < 1$$

absolutely convergent

11. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^3}+\frac{1}{3^n}\right)$ $= \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} + \sum_{n=1}^{\infty} \frac{1}{3^n}$

Convergent

$\frac{1}{n^3}$: p-series, $3 > 1$, Convergent

$\frac{1}{3^n} = \left(\frac{1}{3}\right)^n$, $r = \frac{1}{3} < 1$, Convergent

1. If the series is of the form $\sum 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leqslant 1$.

2. If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geqslant 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.

14. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$

Converges

$$\left|\frac{\sin 2n}{1+2^n}\right| \leq \frac{1}{1+2^n} < \frac{1}{2^n}$$

$\left(\frac{1}{2}\right)^n$

$r = 1/2 < 1$

converges

The Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\sum a_n$ is also divergent.

16. $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n}$ Diverges

$b_n = \frac{1}{n}$ diverges

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n^4+1}}{n^3+n}}{1/n} \quad \frac{\cancel{n}\sqrt{n^4+1}}{\cancel{n}(n^2+1)} \quad \frac{\frac{\sqrt{n^4+1}}{n^2+1}}{n^2} \quad \frac{\sqrt{1+1/n^4}}{1+1/n^2} = 1 > 0$$

The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

$$19. \sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$$

$$f'(x) = \frac{2 - \ln x}{2x^{3/2}} < 0$$

$$2 - \ln x < 0$$

$$\ln x > 2 \qquad \boxed{x > e^2}$$

Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} \overset{H}{=\!\!>} \frac{1/n}{1/2\sqrt{n}} = \frac{2\sqrt{n}}{n} = \frac{2}{\sqrt{n}} \to 0$$

satisfies

(i) $b_{n+1} \leqslant b_n$ for all n ✓

(ii) $\lim_{n\to\infty} b_n = 0$ ✓

then the series is convergent.

32. $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$

$$\sqrt[n]{\left(\frac{n!}{n^4}\right)^n} = \lim_{n\to\infty} \frac{n!}{n^4} \to \infty$$

Divergent

The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2 n}$ diverges

$$\frac{1}{n+n\cos^2 n} \geq \frac{1}{n+n} = \frac{1}{2n}$$

$$0 \leq n\cos^2 n \leq n$$

$$\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges