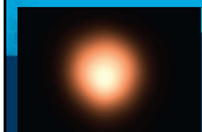

Chapter 11

Infinite Sequences and Series

CENGAGE

1

11.5 Alternating Series

CENGAGE

2

Alternating Series (1 of 3)

In this section we learn how to deal with series whose terms are not necessarily positive. Of particular importance are *alternating series,* whose terms alternate in sign.

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$

$$-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\frac{5}{6}+\frac{6}{7}-\cdots=\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+1}$$

CENGAGE

3

Alternating Series (2 of 3)

We see from these examples that the *n*th term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n \quad \text{or} \quad a_n = (-1)^n b_n$$

where b_n is a positive number. (In fact, $b_n = |a_n|$.)

The following test says that if the terms of an alternating series decrease toward 0 in absolute value, then the series converges.

CENGAGE

4

Alternating Series (3 of 3)

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \quad b_n > 0$$

satisfies

(i) $b_{n+1} \le b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

CENGAGE

5

Example 1

The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies

(i) $b_{n+1} < b_n$ because $\frac{1}{n+1} < \frac{1}{n}$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

so the series is convergent by the Alternating Series Test.

CENGAGE

6

11.6 Absolute Convergence and the Ratio and Root Tests

7

Absolute Convergence and the Ratio and Root Tests (1 of 9)

Given any series $\sum a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute values of the terms of the original series.

1 Definition A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

8

Absolute Convergence and the Ratio and Root Tests (2 of 9)

Notice that if $\sum a_n$ is a series with positive terms, then $|a_n| = a_n$ and so absolute convergence is the same as convergence in this case.

9

Example 1

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is a convergent *p*-series ($p = 2$).

CENGAGE

10

Example 2

We know that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is convergent, but it is not absolutely convergent because the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

which is the harmonic series (*p*-series with $p = 1$) and is therefore divergent.

CENGAGE

11

Absolute Convergence and the Ratio and Root Tests (3 of 9)

2 Definition A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Example 2 shows that the alternating harmonic series is conditionally convergent. Thus it is possible for a series to be convergent but not absolutely convergent.

However, the next theorem shows that absolute convergence implies convergence.

3 Theorem If a series $\sum a_n$ is absolutely convergent, then it is convergent.

CENGAGE

12

Example 3

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 1}{3^2} + \cdots$$

is convergent or divergent.

Solution:

This series has both positive and negative terms, but it is not alternating. (The first term is positive, the next three are negative, and the following three are positive: The signs change irregularly.)

CENGAGE

13

Example 3 – *Solution* (1 of 2)

We can apply the Comparison Test to the series of absolute values

$$\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

Since $|\cos n| \le 1$ for all n, we have

$$\frac{|\cos n|}{n^2} \le \frac{1}{n^2}$$

We know that $\sum \frac{1}{n^2}$ is convergent (p-series with $p = 2$) and therefore $\sum \frac{|\cos n|}{n^2}$ is convergent by the Comparison Test.

CENGAGE

14

Example 3 – *Solution* (2 of 2)

Thus the given series $\sum \frac{|\cos n|}{n^2}$ is absolutely convergent and therefore convergent by Theorem 3.

CENGAGE

15

Absolute Convergence and the Ratio and Root Tests (4 of 9)

The following test is very useful in determining whether a given series is absolutely convergent.

The Ratio Test

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

CENGAGE

16

Absolute Convergence and the Ratio and Root Tests (5 of 9)

Note:

Part (iii) of the Ratio Test says that if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the test gives no information.

For instance, for the convergent series $\sum \frac{1}{n^2}$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \frac{1}{\left(1+\frac{1}{n}\right)^2} \to 1 \quad \text{as } n \to \infty$$

CENGAGE

17

Absolute Convergence and the Ratio and Root Tests (6 of 9)

whereas for the divergent series $\sum \frac{1}{n}$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to 1 \quad \text{as } n \to \infty$$

Therefore, if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the series $\sum a_n$ might converge or it might diverge.

In this case the Ratio Test fails and we must use some other test.

CENGAGE

18

Example 5

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

Solution:
Since the terms $a_n = \frac{n^n}{n!}$ are positive, we don't need the absolute value signs.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$$
$$= \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n}$$

CENGAGE

19

Example 5 – *Solution*

$$= \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \to e \quad \text{as } n \to \infty$$

Since $e > 1$, the given series is divergent by the Ratio Test.

CENGAGE

20

Absolute Convergence and the Ratio and Root Tests (7 of 9)

Note:
Although the Ratio Test works in Example 5, an easier method is to use the Test for Divergence. Since

$$a_n = \frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n} \geq n$$

it follows that a_n does not approach 0 as $n \to \infty$. Therefore the given series is divergent by the Test for Divergence.

CENGAGE

21

Absolute Convergence and the Ratio and Root Tests (8 of 9)

The following test is convenient to apply when *n*th powers occur.

The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

22

Absolute Convergence and the Ratio and Root Tests (9 of 9)

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then part (iii) of the Root Test says that the test gives no information. The series $\sum a_n$ could converge or diverge.

(If *L* = 1 in the Ratio Test, don't try the Root Test because *L* will again be 1. And if *L* = 1 in the Root Test, don't try the Ratio Test because it will fail too.)

23

Example 6

Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

Solution:

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2+\frac{3}{n}}{3+\frac{2}{n}} \to \frac{2}{3} < 1$$

Thus the given series is absolutely convergent (and therefore convergent) by the Root Test.

24