

12.3 The Dot Product

1 Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of \mathbf{a} and \mathbf{b} is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

2 Properties of the Dot Product If \mathbf{a} , \mathbf{b} , and \mathbf{c} are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$

5. $\mathbf{0} \cdot \mathbf{a} = 0$

EXAMPLE 1

$$\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 2(3) + 4(-1) = \textcircled{2}$$

$$\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\frac{1}{2} \rangle = -1(6) + 7(2) + 4(-\frac{1}{2}) = \textcircled{6}$$

$$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k}) = 1(0) + 2(2) + (-3)(-1) = \textcircled{7}$$

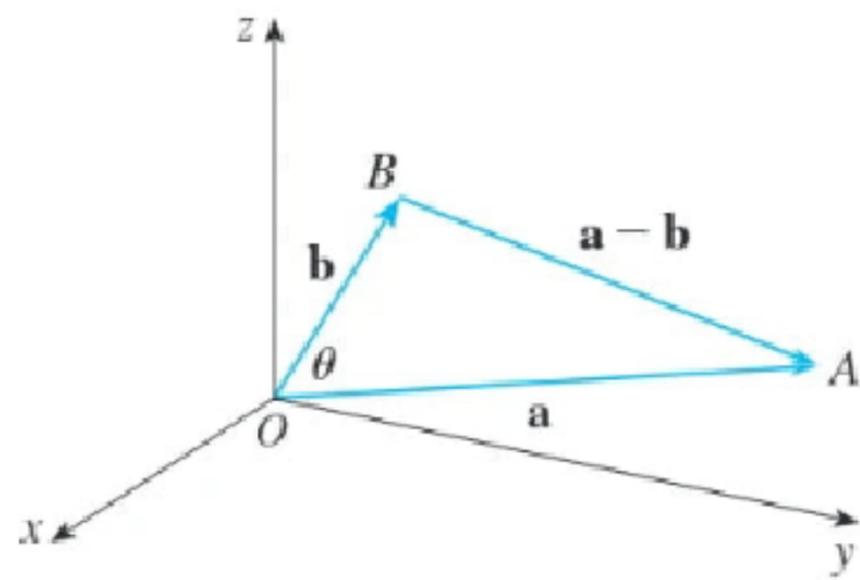


FIGURE 1

The dot product $\mathbf{a} \cdot \mathbf{b}$ can be given a geometric interpretation in terms of the **angle θ between \mathbf{a} and \mathbf{b}** , which is defined to be the angle between the representations of \mathbf{a} and \mathbf{b} that start at the origin, where $0 \leq \theta \leq \pi$. In other words, θ is the angle between the line segments \overrightarrow{OA} and \overrightarrow{OB} in Figure 1. Note that if \mathbf{a} and \mathbf{b} are parallel vectors, then $\theta = 0$ or $\theta = \pi$.

The formula in the following theorem is used by physicists as the *definition* of the dot product.

3 Theorem If θ is the angle between the vectors \mathbf{a} and \mathbf{b} , then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

EXAMPLE 2 If the vectors \mathbf{a} and \mathbf{b} have lengths 4 and 6, and the angle between them is $\pi/3$, find $\mathbf{a} \cdot \mathbf{b}$.

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

$$\mathbf{a} \cdot \mathbf{b} = (4)(6) \cos \pi/3$$

$$\mathbf{a} \cdot \mathbf{b} = (4)(6) \left(\frac{1}{2} \right)$$

$$\mathbf{a} \cdot \mathbf{b} = 12$$

6 Corollary If θ is the angle between the nonzero vectors \mathbf{a} and \mathbf{b} , then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$$

EXAMPLE 3 Find the angle between the vectors $\mathbf{a} = \langle 2, 2, -1 \rangle$ and $\mathbf{b} = \langle 5, -3, 2 \rangle$.

$$|\mathbf{a}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{9} = 3$$

$$|\mathbf{b}| = \sqrt{(5)^2 + (-3)^2 + (2)^2} = \sqrt{38}$$

$$\mathbf{a} \cdot \mathbf{b} = 2(5) + 2(-3) + (-1)(2) = 2$$

$$\cos \theta = \frac{2}{3\sqrt{38}} \Rightarrow \theta = 83.8^\circ$$

Two nonzero vectors \mathbf{a} and \mathbf{b} are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \pi/2$. Then Theorem 3 gives

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos(\pi/2) = 0$$

and conversely if $\mathbf{a} \cdot \mathbf{b} = 0$, then $\cos \theta = 0$, so $\theta = \pi/2$. The zero vector $\mathbf{0}$ is considered to be perpendicular to all vectors. Therefore we have the following method for determining whether two vectors are orthogonal.

7 Two vectors \mathbf{a} and \mathbf{b} are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

EXAMPLE 4 Show that $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is perpendicular to $5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (2)(5) + (2)(-4) + (-1)(2) \\ &= 10 - 8 - 2 \end{aligned}$$

$$\mathbf{a} \cdot \mathbf{b} = 0$$

perpendicular

Direction Angles and Direction Cosines

The **direction angles** of a nonzero vector \mathbf{a} are the angles α , β , and γ (in the interval $[0, \pi]$) that \mathbf{a} makes with the positive x -, y -, and z -axes, respectively. (See Figure 3.)

The cosines of these direction angles, $\cos \alpha$, $\cos \beta$, and $\cos \gamma$, are called the **direction cosines** of the vector \mathbf{a} . Using Corollary 6 with \mathbf{b} replaced by

$$\boxed{8} \quad \cos \alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{|\mathbf{a}| |\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}$$

(This can also be seen directly from Figure 3.)

Similarly, we also have

$$\boxed{9} \quad \cos \beta = \frac{a_2}{|\mathbf{a}|} \quad \cos \gamma = \frac{a_3}{|\mathbf{a}|}$$

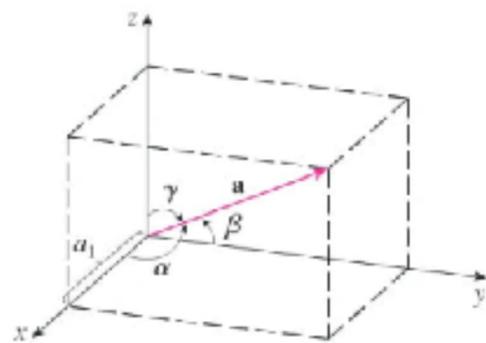


FIGURE 3

By squaring the expressions in Equations 8 and 9 and adding, we see that

$$\boxed{10} \quad \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

We can also use Equations 8 and 9 to write

$$\begin{aligned} \mathbf{a} &= \langle a_1, a_2, a_3 \rangle = \langle |\mathbf{a}| \cos \alpha, |\mathbf{a}| \cos \beta, |\mathbf{a}| \cos \gamma \rangle \\ &= |\mathbf{a}| \langle \cos \alpha, \cos \beta, \cos \gamma \rangle \end{aligned}$$

Therefore

$$\boxed{11} \quad \frac{1}{|\mathbf{a}|} \mathbf{a} = \langle \cos \alpha, \cos \beta, \cos \gamma \rangle$$

which says that the direction cosines of \mathbf{a} are the components of the unit vector in the direction of \mathbf{a} .

EXAMPLE 5 Find the direction angles of the vector $\mathbf{a} = \langle 1, 2, 3 \rangle$.

$$|\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\cos \alpha = \frac{1}{\sqrt{14}} = 74.5^\circ$$

$$\cos \beta = \frac{2}{\sqrt{14}} = 57.7^\circ$$

$$\cos \gamma = \frac{3}{\sqrt{14}} = 36.7^\circ$$

$$\left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$$

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$$W = |\mathbf{F}| |\mathbf{D}| \cos \theta = \mathbf{F} \cdot \mathbf{D}$$

Thus the work done by a constant force \mathbf{F} is the dot product $\mathbf{F} \cdot \mathbf{D}$, where \mathbf{D} is the displacement vector.

EXAMPLE 7 A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal. Find the work done by the force.

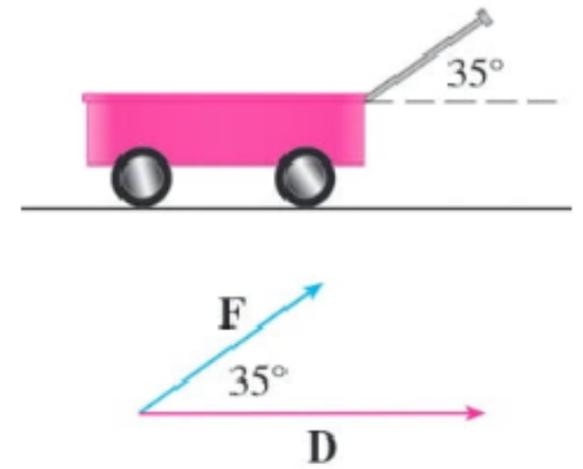


FIGURE 7

$$\begin{aligned} W &= |F| |D| \cos \theta \\ &= (70)(100) \cos 35 \\ &= 5734 \text{ J} \end{aligned}$$

EXAMPLE 8 A force is given by a vector $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and moves a particle from the point $P(2, 1, 0)$ to the point $Q(4, 6, 2)$. Find the work done.

$$D = \vec{PQ} = \langle 4-2, 6-1, 2-0 \rangle = \langle 2, 5, 2 \rangle$$

$$W = \mathbf{F} \cdot D = (3)(2) + (4)(5) + (5)(2) \\ = \boxed{36 \text{ J}}$$

12.4 The Cross Product

4 Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of \mathbf{a} and \mathbf{b} is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

Notice that the **cross product** $\mathbf{a} \times \mathbf{b}$ of two vectors \mathbf{a} and \mathbf{b} , unlike the dot product, is a vector. For this reason it is also called the **vector product**. Note that $\mathbf{a} \times \mathbf{b}$ is defined only when \mathbf{a} and \mathbf{b} are *three-dimensional* vectors.

$$\mathbf{6} \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

In view of the similarity between Equations 5 and 6, we often write

$$\mathbf{7} \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

EXAMPLE 1 If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$\begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k}$$

$$(-15 - 28) \mathbf{i} - (-5 - 8) \mathbf{j} + (7 - 6) \mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = -43 \mathbf{i} + 13 \mathbf{j} + \mathbf{k}$$

8 Theorem The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b} .

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3$$

$$= a_1 (a_2 b_3 - a_3 b_2) - a_2 (a_1 b_3 - a_3 b_1) + a_3 (a_1 b_2 - a_2 b_1)$$

$$= \cancel{a_1 a_2 b_3} - \cancel{a_1 a_3 b_2} - \cancel{a_2 a_1 b_3} + \cancel{a_2 a_3 b_1} + \cancel{a_3 a_1 b_2} - \cancel{a_3 a_2 b_1}$$

$$= 0$$

9 Theorem If θ is the angle between \mathbf{a} and \mathbf{b} (so $0 \leq \theta \leq \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$$

10 Corollary Two nonzero vectors \mathbf{a} and \mathbf{b} are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

EXAMPLE 3 Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

$$\vec{PQ} = \langle -3, 1, -7 \rangle \quad \vec{PR} = \langle 0, -5, -5 \rangle$$

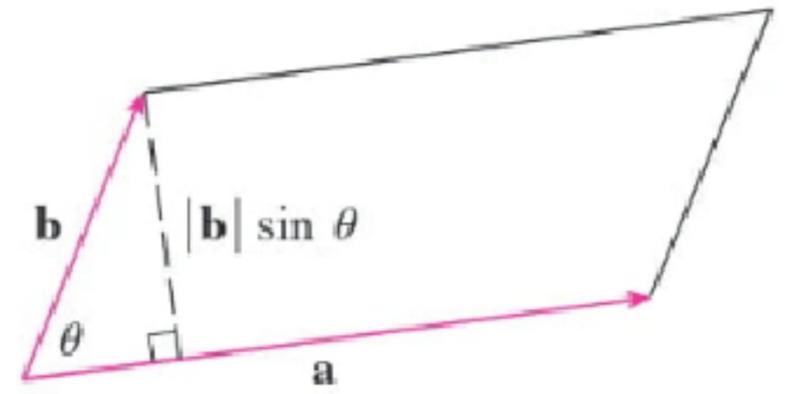
$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -7 \\ -5 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & -7 \\ 0 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 1 \\ 0 & -5 \end{vmatrix} \mathbf{k}$$

$$= (-5 - 35)\mathbf{i} - (15 - 0)\mathbf{j} + (15 - 0)\mathbf{k}$$

$$= \boxed{-40\mathbf{i} - 15\mathbf{j} + 15\mathbf{k} \quad \langle -40, -15, 15 \rangle}$$

The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by \mathbf{a} and \mathbf{b} .



EXAMPLE 4 Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

FIGURE 2

$$|\mathbf{a} \times \mathbf{b}| = \underbrace{|\mathbf{a}|}_{\text{base}} \underbrace{|\mathbf{b}| \sin \theta}_{\text{height}}$$

$$\langle -40, -15, 15 \rangle$$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{(-40)^2 + (-15)^2 + (15)^2} = \sqrt{1660} = 5\sqrt{82}$$

$$\text{Triangle Area is } \frac{5}{2} \sqrt{82}$$

11 Properties of the Cross Product If \mathbf{a} , \mathbf{b} , and \mathbf{c} are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$

3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

■ Triple Products

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ that occurs in Property 5 is called the **scalar triple product** of the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} . Notice from Equation 12 that we can write the scalar triple product as a determinant:

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$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

EXAMPLE 5 Use the scalar triple product to show that the vectors $\mathbf{a} = \langle 1, 4, -7 \rangle$, $\mathbf{b} = \langle 2, -1, 4 \rangle$, and $\mathbf{c} = \langle 0, -9, 18 \rangle$ are coplanar.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} (1) - \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} (4) + \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix} (-7)$$

$$= (-18 - (-36)) - 4(36) - 7(-18) = \boxed{0}$$

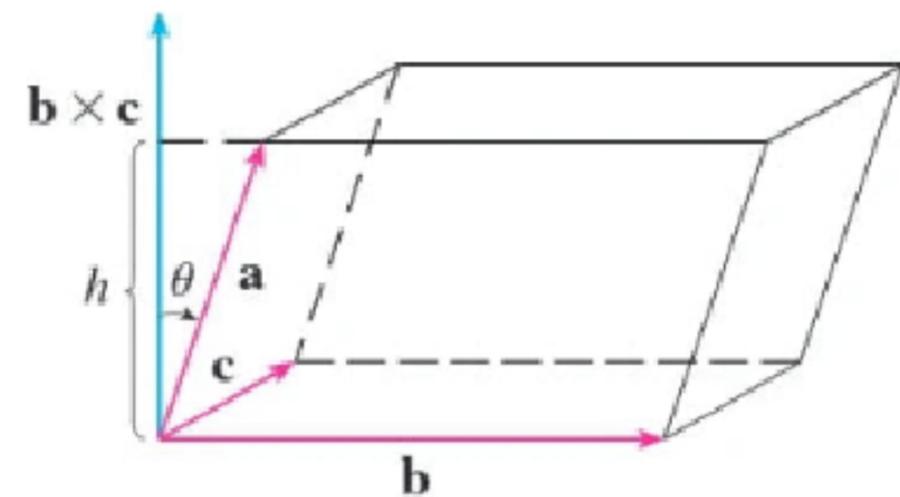


FIGURE 3

EXAMPLE 6 A bolt is tightened by applying a 40-N force to a 0.25-m wrench as shown in Figure 5. Find the magnitude of the torque about the center of the bolt.

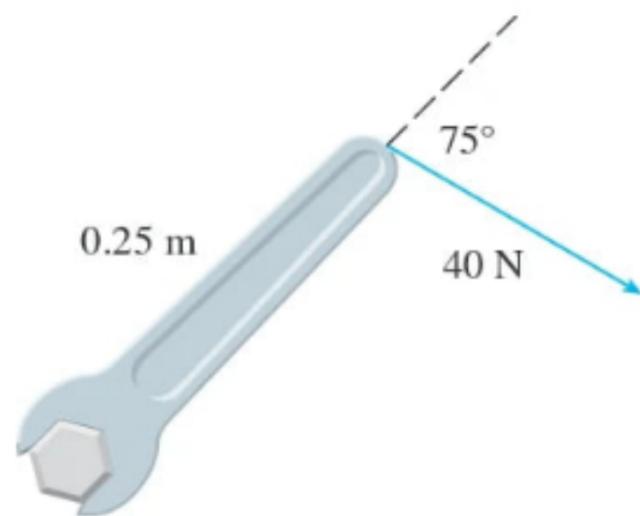


FIGURE 5

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}| |\mathbf{F}| \sin \theta$$

$$|\tau| = |\mathbf{r} \times \mathbf{F}|$$

$$|\mathbf{F}| = 40 \text{ N}$$

$$|\mathbf{r}| = 0.25 \text{ m}$$

$$\theta = 75^\circ$$

$$|\tau| = (0.25)(40) \sin 75^\circ$$

$$= \boxed{9.66 \text{ N}\cdot\text{m}}$$