

Chapter 12
Vectors and the Geometry of Space

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12.4 The Cross Product

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The Cross Product (1 of 19)

Given two nonzero vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, it is very useful to be able to find a nonzero vector \mathbf{c} that is perpendicular to both \mathbf{a} and \mathbf{b} .

If $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ is such a vector, then $\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 0$ and so

- 1 $a_1c_1 + a_2c_2 + a_3c_3 = 0$
- 2 $b_1c_1 + b_2c_2 + b_3c_3 = 0$

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The Cross Product (2 of 19)

To eliminate c_3 we multiply (1) by b_3 and (2) by a_3 and subtract:

$$3 \quad (a_3b_3 - a_3b_3)c_1 + (a_2b_3 - a_3b_2)c_2 = 0$$

Equation 3 has the form $pc_1 + qc_2 = 0$, for which an obvious solution is $c_1 = q$ and $c_2 = -p$. So a solution of (3) is

$$c_1 = a_2b_3 - a_3b_2 \quad c_2 = a_3b_1 - a_1b_3$$



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The Cross Product (3 of 19)

Substituting these values into (1) and (2), we then get

$$c_3 = a_1b_2 - a_2b_1$$

This means that a vector perpendicular to both \mathbf{a} and \mathbf{b} is

$$\langle c_1, c_2, c_3 \rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

The resulting vector is called the *cross product* of \mathbf{a} and \mathbf{b} and is denoted by $\mathbf{a} \times \mathbf{b}$.



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The Cross Product (4 of 19)

4 Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of \mathbf{a} and \mathbf{b} is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

Notice that the **cross product** $\mathbf{a} \times \mathbf{b}$ of two vectors \mathbf{a} and \mathbf{b} , unlike the dot product, is a vector. For this reason it is also called the **vector product**.

Note that $\mathbf{a} \times \mathbf{b}$ is defined only when \mathbf{a} and \mathbf{b} are *three-dimensional* vectors.



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The Cross Product (5 of 19)

In order to make Definition 4 easier to remember, we use the notation of determinants.

A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example,

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14$$



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The Cross Product (6 of 19)

A **determinant of order 3** can be defined in terms of second-order determinants as follows:

$$5 \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Observe that each term on the right side of Equation 5 involves a number a_i in the first row of the determinant, and a_i is multiplied by the second-order determinant obtained from the left side by deleting the row and column in which a_i appears.



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The Cross Product (7 of 19)

Notice also the minus sign in the second term. For example,

$$\begin{aligned} \begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} &= 1 \begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix} \\ &= 1(0 - 4) - 2(6 + 5) + (-1)(12 - 0) \\ &= -38 \end{aligned}$$



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The Cross Product (8 of 19)

If we now rewrite Definition 4 using second-order determinants and the standard basis vectors \mathbf{i} , \mathbf{j} , and \mathbf{k} , we see that the cross product of the vectors $\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$ and $\mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}$ is

$$6 \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$



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The Cross Product (9 of 19)

In view of the similarity between Equations 5 and 6, we often write

$$7 \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Although the first row of the symbolic determinant in Equation 7 consists of vectors, if we expand it as if it were an ordinary determinant using the rule in Equation 5, we obtain Equation 6.

The symbolic formula in Equation 7 is probably the easiest way of remembering and computing cross products.



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Example 1

If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$, then

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} \\ &= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k} \\ &= (-15 - 28)\mathbf{i} - (-5 - 8)\mathbf{j} + (7 - 6)\mathbf{k} \\ &= -43\mathbf{i} + 13\mathbf{j} + \mathbf{k} \end{aligned}$$



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The Cross Product (10 of 19)

We constructed the cross product $\mathbf{a} \times \mathbf{b}$ so that it would be perpendicular to both \mathbf{a} and \mathbf{b} . This is one of the most important properties of a cross product.

8 Theorem The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b} .

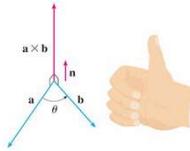


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The Cross Product (11 of 19)

If \mathbf{a} and \mathbf{b} are represented by directed line segments with the same initial point (as in Figure 1), then Theorem 8 says that the cross product $\mathbf{a} \times \mathbf{b}$ points in a direction perpendicular to the plane through \mathbf{a} and \mathbf{b} .



The right-hand rule gives the direction of $\mathbf{a} \times \mathbf{b}$.
Figure 1



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The Cross Product (12 of 19)

It turns out that the direction of $\mathbf{a} \times \mathbf{b}$ is given by the *right-hand rule*: If the fingers of your right hand curl in the direction of a rotation (through an angle less than 180°) from \mathbf{a} to \mathbf{b} , then your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

Now that we know the direction of the vector $\mathbf{a} \times \mathbf{b}$, the remaining thing we need to complete its geometric description is its length $|\mathbf{a} \times \mathbf{b}|$. This is given by following theorem.

9 Theorem If θ is the angle between \mathbf{a} and \mathbf{b} (so $0 \leq \theta \leq \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin \theta$$



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The Cross Product (13 of 19)

Since a vector is completely determined by its magnitude and direction, we can now say that $\mathbf{a} \times \mathbf{b}$ is the vector that is perpendicular to both \mathbf{a} and \mathbf{b} , whose orientation is determined by the right-hand rule, and whose length is $|\mathbf{a}||\mathbf{b}|\sin\theta$. In fact, that is exactly how physicists *define* $\mathbf{a} \times \mathbf{b}$.

10 Corollary Two nonzero vectors \mathbf{a} and \mathbf{b} are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$



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The Cross Product (14 of 19)

The geometric interpretation of Theorem 9 can be seen by looking at Figure 2.

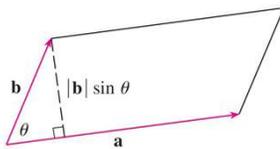


Figure 2



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The Cross Product (15 of 19)

If \mathbf{a} and \mathbf{b} are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\mathbf{a}|$, altitude $|\mathbf{b}|\sin\theta$, and area

$$A = |\mathbf{a}|(|\mathbf{b}|\sin\theta) = |\mathbf{a} \times \mathbf{b}|$$

Thus we have the following way of interpreting the magnitude of a cross product.

The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by \mathbf{a} and \mathbf{b} .



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Example 4

Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Solution:

We computed that $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -40, -15, 15 \rangle$. The area of the parallelogram with adjacent sides PQ and PR is the length of this cross product:

$$\begin{aligned} |\overrightarrow{PQ} \times \overrightarrow{PR}| &= \sqrt{(-40)^2 + (-15)^2 + 15^2} \\ &= 5\sqrt{82} \end{aligned}$$

The area A of the triangle PQR is half the area of this parallelogram, that is,

$$\frac{5}{2}\sqrt{82}.$$



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The Cross Product (16 of 19)

If we apply Theorems 8 and 9 to the standard basis vectors \mathbf{i} , \mathbf{j} , and \mathbf{k} using $\theta = \frac{\pi}{2}$, we obtain

$$\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}$$

Observe that

$$\mathbf{i} \times \mathbf{j} \neq \mathbf{j} \times \mathbf{i}$$



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The Cross Product (17 of 19)

Thus the cross product is not commutative. Also

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

whereas

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$$

So the associative law for multiplication does not usually hold; that is, in general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

However, some of the usual laws of algebra *do* hold for cross products.



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The Cross Product (18 of 19)

The following theorem summarizes the properties of vector products.

11 Properties of the Cross Product If \mathbf{a} , \mathbf{b} , and \mathbf{c} are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$



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The Cross Product (19 of 19)

These properties can be proved by writing the vectors in terms of their components and using the definition of a cross product.

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, then

$$\begin{aligned}
 \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \\
 &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \\
 &= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3 \\
 &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}
 \end{aligned}$$



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Triple Products



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Triple Products (1 of 5)

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ that occurs in Property 5 is called the **scalar triple product** of the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} . Notice from Equation 12 that we can write the scalar triple product as a determinant:

$$13 \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



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Triple Products (2 of 5)

The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} . (See Figure 3.)

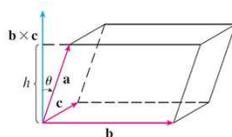


Figure 3

The area of the base parallelogram is $A = |\mathbf{b} \times \mathbf{c}|$.



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Triple Products (3 of 5)

If θ is the angle between \mathbf{a} and $\mathbf{b} \times \mathbf{c}$, then the height h of the parallelepiped is $h = |\mathbf{a}||\cos \theta|$. (We must use $|\cos \theta|$ instead of $\cos \theta$ in case $\theta > \frac{\pi}{2}$.) Therefore the volume of the parallelepiped is

$$V = Ah = |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos \theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Thus we have proved the following formula.

14 The volume of the parallelepiped determined by the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$



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Triple Products (4 of 5)

If we use the formula in (14) and discover that the volume of the parallelepiped determined by \mathbf{a} , \mathbf{b} , and \mathbf{c} is 0, then the vectors must lie in the same plane; that is, they are **coplanar**.



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Example 5

Use the scalar triple product to show that the vectors $\mathbf{a} = \langle 1, 4, -7 \rangle$, $\mathbf{b} = \langle 2, -1, 4 \rangle$, and $\mathbf{c} = \langle 0, -9, 18 \rangle$, are coplanar.

Solution:

We use Equation 13 to compute their scalar triple product:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$



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Example 5 – Solution

$$\begin{aligned} &= 1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7 \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix} \\ &= 1(18) - 4(36) - 7(-18) \\ &= 0 \end{aligned}$$

Therefore, by (14), the volume of the parallelepiped determined by \mathbf{a} , \mathbf{b} , and \mathbf{c} is 0. This means that \mathbf{a} , \mathbf{b} , and \mathbf{c} are coplanar.



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Triple Products (5 of 5)

The product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ that occurs in Property 6 is called the **vector triple product** of \mathbf{a} , \mathbf{b} , and \mathbf{c} .



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Torque



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Torque (1 of 3)

The idea of a cross product occurs often in physics. In particular, we consider a force \mathbf{F} acting on a rigid body at a point given by a position vector \mathbf{r} . (For instance, if we tighten a bolt by applying a force to a wrench as in Figure 4, we produce a turning effect.)

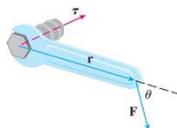


Figure 4



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Torque (2 of 3)

The **torque** τ (relative to the origin) is defined to be the cross product of the position and force vectors

$$\tau = \mathbf{r} \times \mathbf{F}$$

and measures the tendency of the body to rotate about the origin. The direction of the torque vector indicates the axis of rotation.



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Torque (3 of 3)

According to Theorem 9, the magnitude of the torque vector is

$$|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}|\sin\theta$$

where θ is the angle between the position and force vectors. Observe that the only component of \mathbf{F} that can cause a rotation is the one perpendicular to \mathbf{r} , that is, $|\mathbf{F}|\sin\theta$.

The magnitude of the torque is equal to the area of the parallelogram determined by \mathbf{r} and \mathbf{F} .



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Example 6

A bolt is tightened by applying a 40-N force to a 0.25-m wrench as shown in Figure 5.

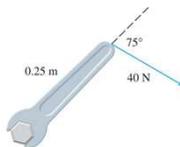


Figure 5

Find the magnitude of the torque about the center of the bolt.



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Example 6 – Solution

The magnitude of the torque vector is

$$\begin{aligned} |\tau| &= |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}| |\mathbf{F}| \sin 75^\circ \\ &= (0.25)(40) \sin 75^\circ \\ &= 10 \sin 75^\circ \approx 9.66 \text{ N} \cdot \text{m} \end{aligned}$$

If the bolt is right-threaded, then the torque vector itself is

$$\tau = |\tau| \mathbf{n} \approx 9.66 \mathbf{n}$$

where \mathbf{n} is a unit vector directed down into the page.



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