Three-Dimensional Coordinate Systems
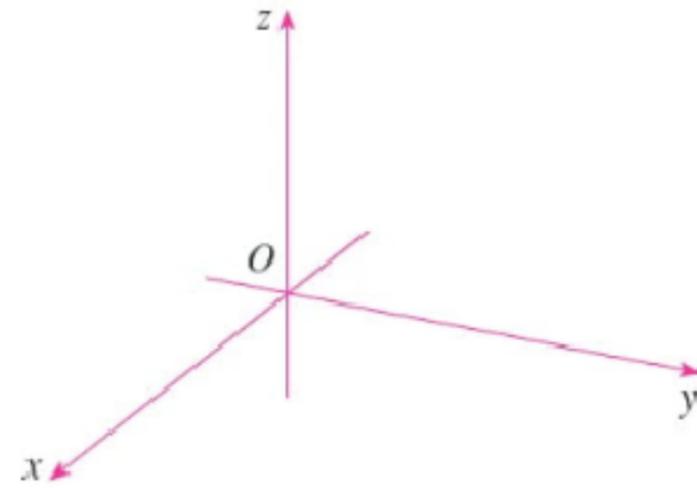

FIGURE 1

Coordinate axes

FIGURE 4

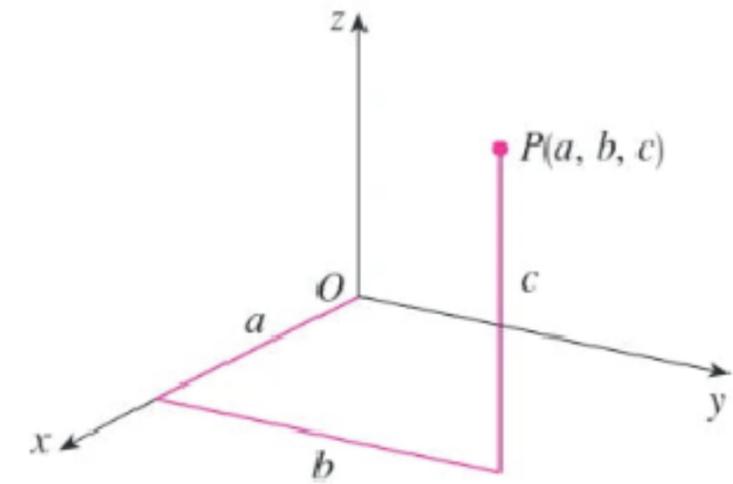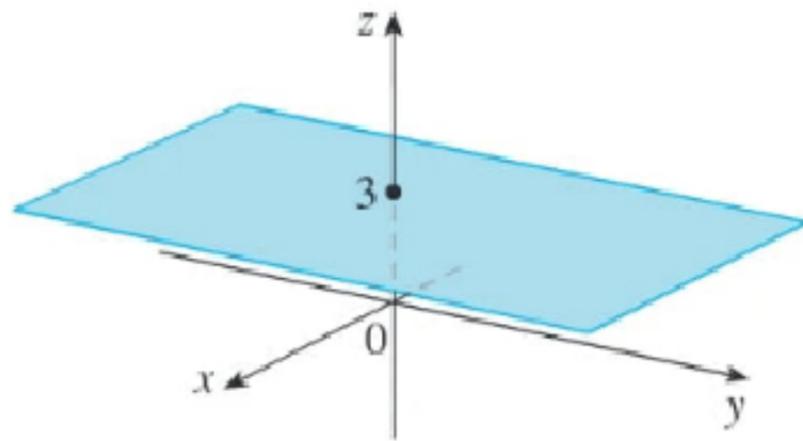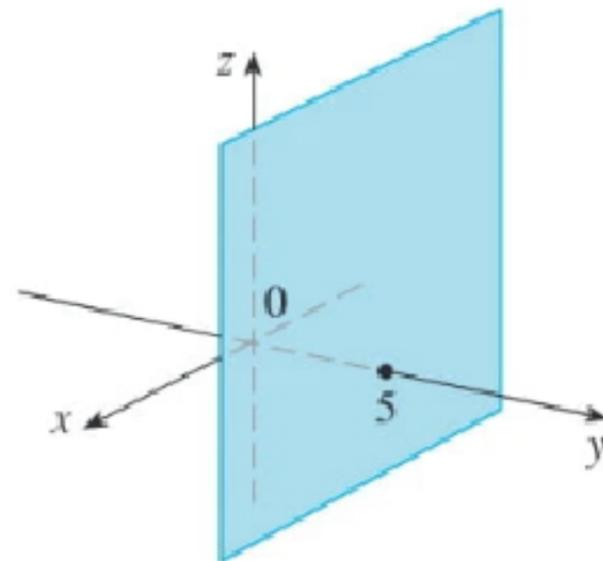**EXAMPLE 1** What surfaces in $\mathbb{R}^3$ are represented by the following equations?

(a) $z = 3$ (b) $y = 5$

(a) $z = 3$, a plane in $\mathbb{R}^3$

(b) $y = 5$, a plane in $\mathbb{R}^3$

EXAMPLE 2

(a) Which points (x, y, z) satisfy the equations

$$x^2 + y^2 = 1 \qquad \text{and} \qquad z = 3$$

(b) What does the equation $x^2 + y^2 = 1$ represent as a surface in $\mathbb{R}^3$?

$$(\mathbb{R} \times \mathbb{R} \times \mathbb{R}) = \mathbb{R}^3$$
$$x \quad y \quad z$$
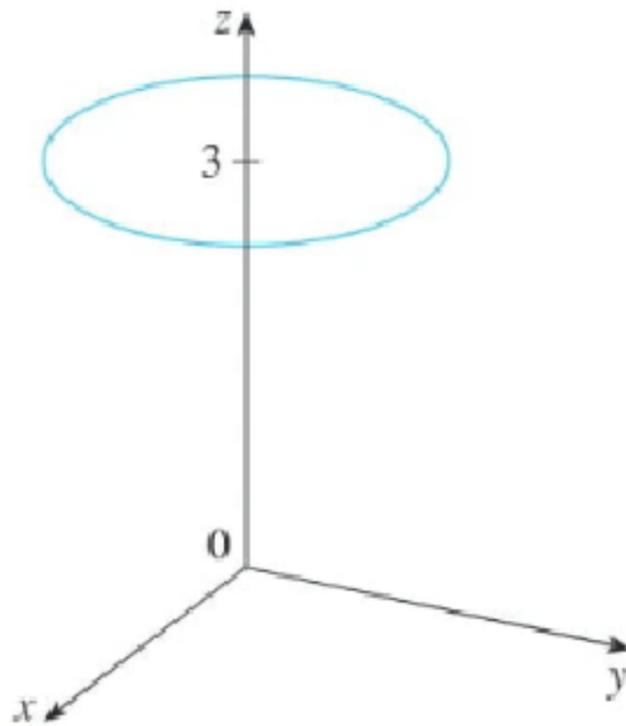
$$\mathbb{R}^2$$

FIGURE 8

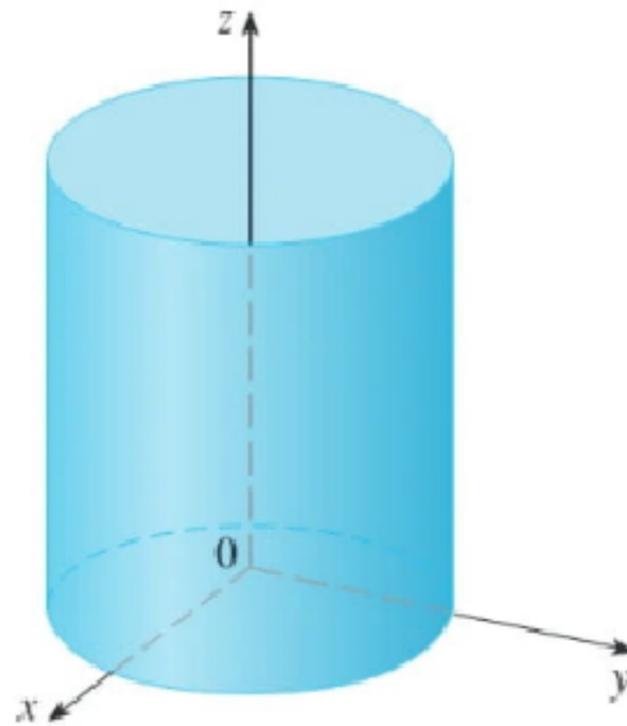The circle $x^2 + y^2 = 1$, $z = 3$

FIGURE 9

The cylinder $x^2 + y^2 = 1$

Distance and Spheres

> **Distance Formula in Three Dimensions** The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is
>
> $$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

EXAMPLE 4 The distance from the point $P(2, -1, 7)$ to the point $Q(1, -3, 5)$:

$$|PQ| = \sqrt{(2-1)^2 + (-1+3)^2 + (7-5)^2}$$
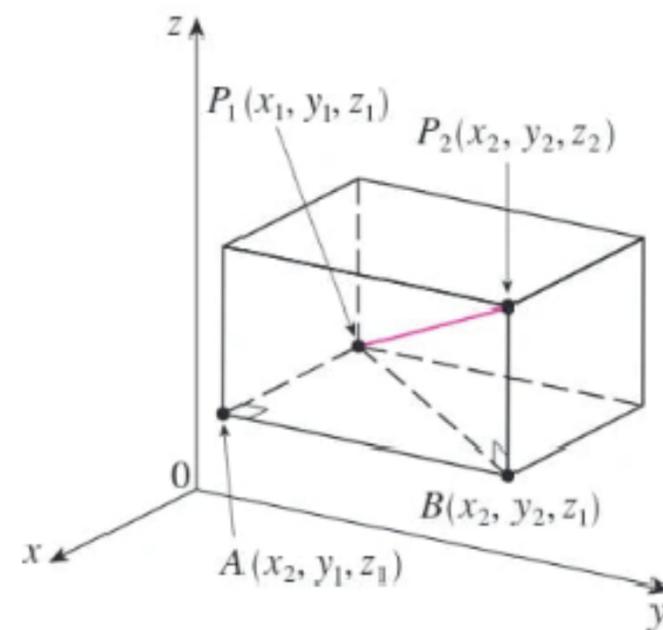
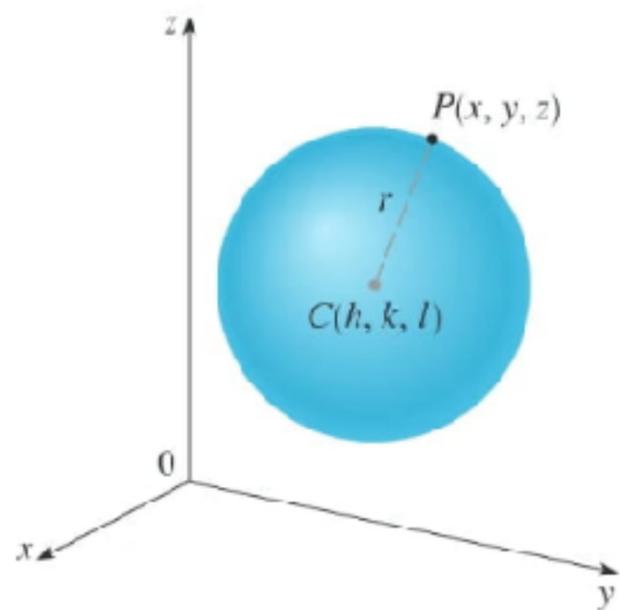$$\sqrt{1+4+4} = \sqrt{9} = \boxed{3}$$

FIGURE 11

FIGURE 12

Equation of a Sphere An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

In particular, if the center is the origin O, then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$

EXAMPLE 6 Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ is the equation of a sphere, and find its center and radius.

$$\left(x^2 + 4x\right) + \left(y^2 - 6y\right) + \left(z^2 + 2z\right) = -6$$
$$+ 4 + 9 + 1$$

$$\left(x^2 + 4x + 4\right) + \left(y^2 - 6y + 9\right) + \left(z^2 + 2z + 1\right)$$

$$(x + 2)^2 + (y - 3)^2 + (z + 1)^2 = 8$$

$$\text{center } (-2, 3, -1) \qquad r = \sqrt{8} = 2\sqrt{2}$$

> **Definition of Vector Addition** If **u** and **v** are vectors positioned so the initial point of **v** is at the terminal point of **u**, then the **sum u + v** is the vector from the initial point of **u** to the terminal point of **v**.

The definition of vector addition is illustrated in Figure 3. You can see why this definition is sometimes called the **Triangle Law**.
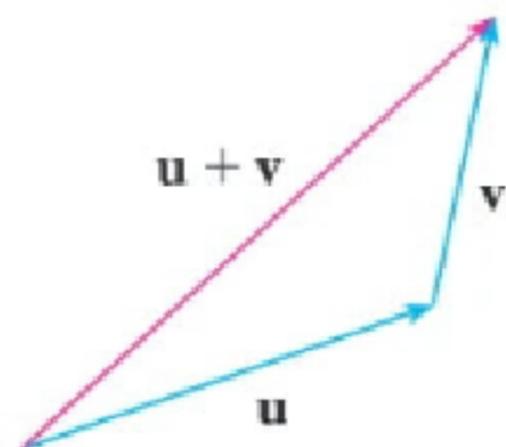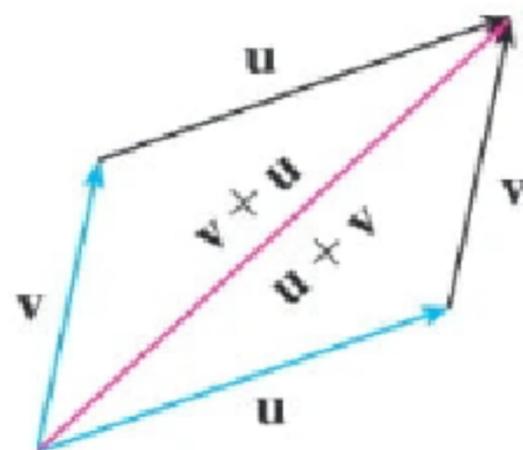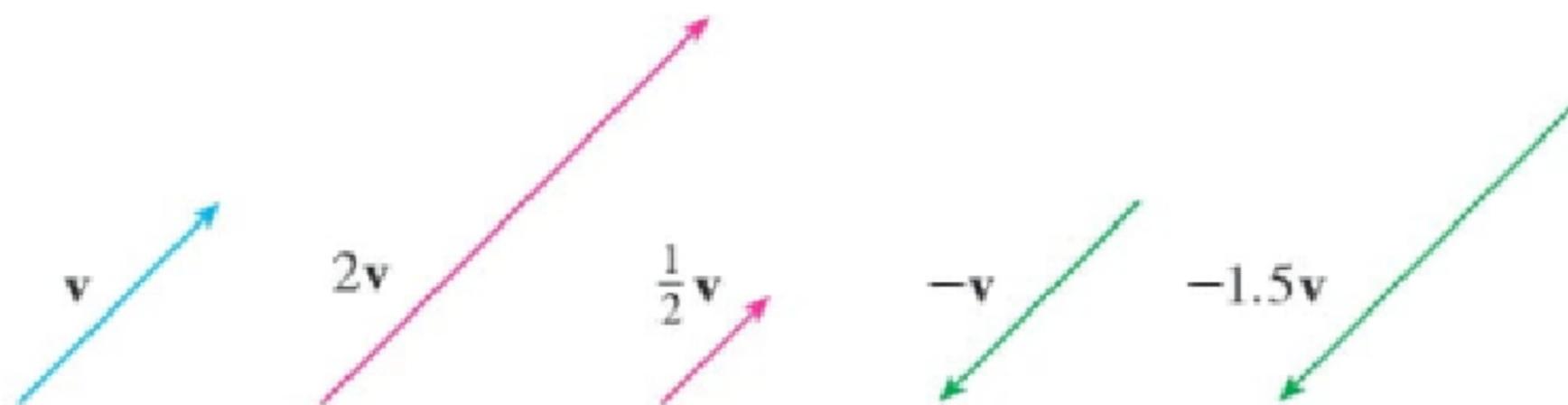
FIGURE 3
The Triangle Law

FIGURE 4
The Parallelogram Law

Definition of Scalar Multiplication If c is a scalar and $\mathbf{v}$ is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

This definition is illustrated in Figure 7. We see that real numbers work like scaling factors here; that's why we call them scalars. Notice that two nonzero vectors are **parallel** if they are scalar multiples of one another. In particular, the vector $-\mathbf{v} = (-1)\mathbf{v}$ has the same length as $\mathbf{v}$ but points in the opposite direction. We call it the **negative** of $\mathbf{v}$.

$\mathbf{v}$ $2\mathbf{v}$ $\frac{1}{2}\mathbf{v}$ $-\mathbf{v}$ $-1.5\mathbf{v}$
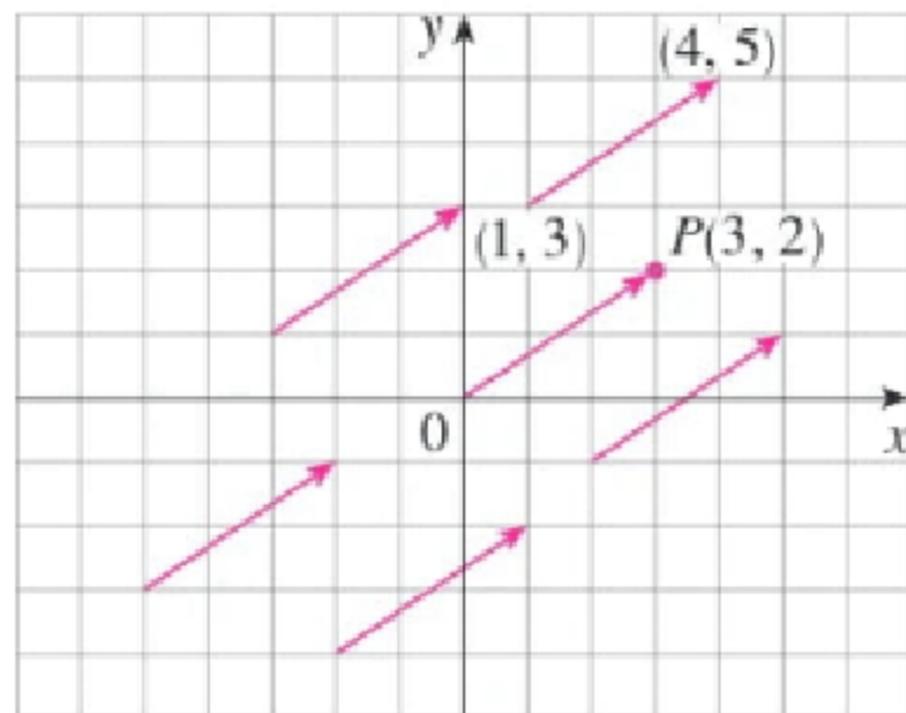
Components

FIGURE 12
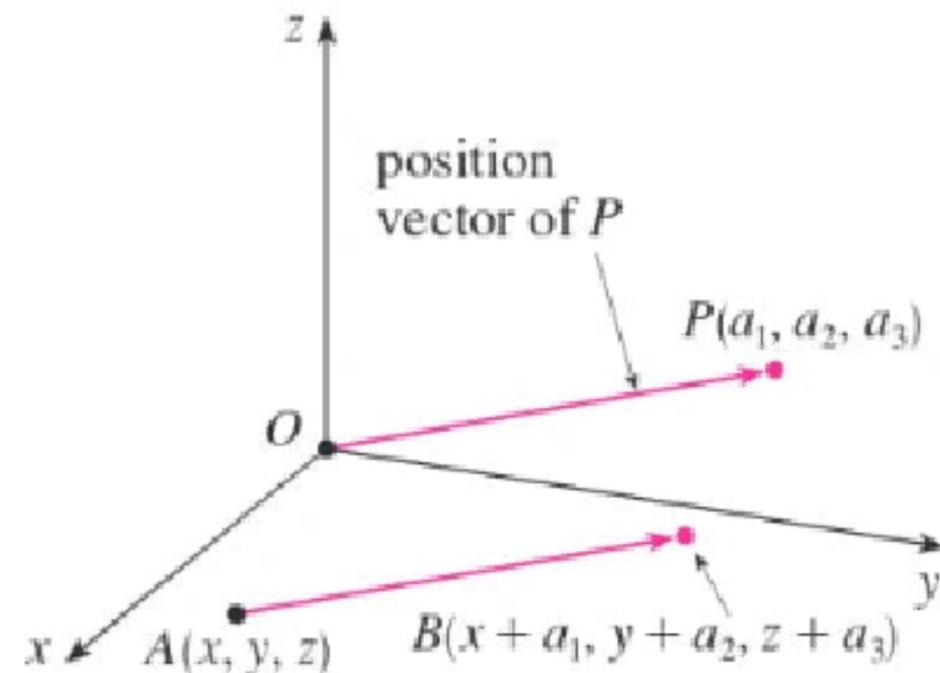Representations of $\mathbf{a} = \langle 3, 2 \rangle$

FIGURE 13
Representations of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

1 Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector **a** with representation $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

EXAMPLE 3 Find the vector represented by the directed line segment with initial point $A(2, -3, 4)$ and terminal point $B(-2, 1, 1)$.

$$A\,(2, -3, 4) \qquad B\,(-2, 1, 1)$$
$$\quad\; x_1 \; y_1 \; z_1 \qquad\qquad\qquad x_2 \; y_2 \; z_2$$

$$\langle -2-2, \; 1--3, \; 1-4 \rangle$$

$$\boxed{\langle -4, \; 4, \; -3 \rangle} \quad \rightarrow \text{Poistion Vector}$$
$$\qquad\qquad\qquad\qquad\qquad \text{Origin} \rightarrow \text{Point}$$

A (5, -2)

B (7, 3)
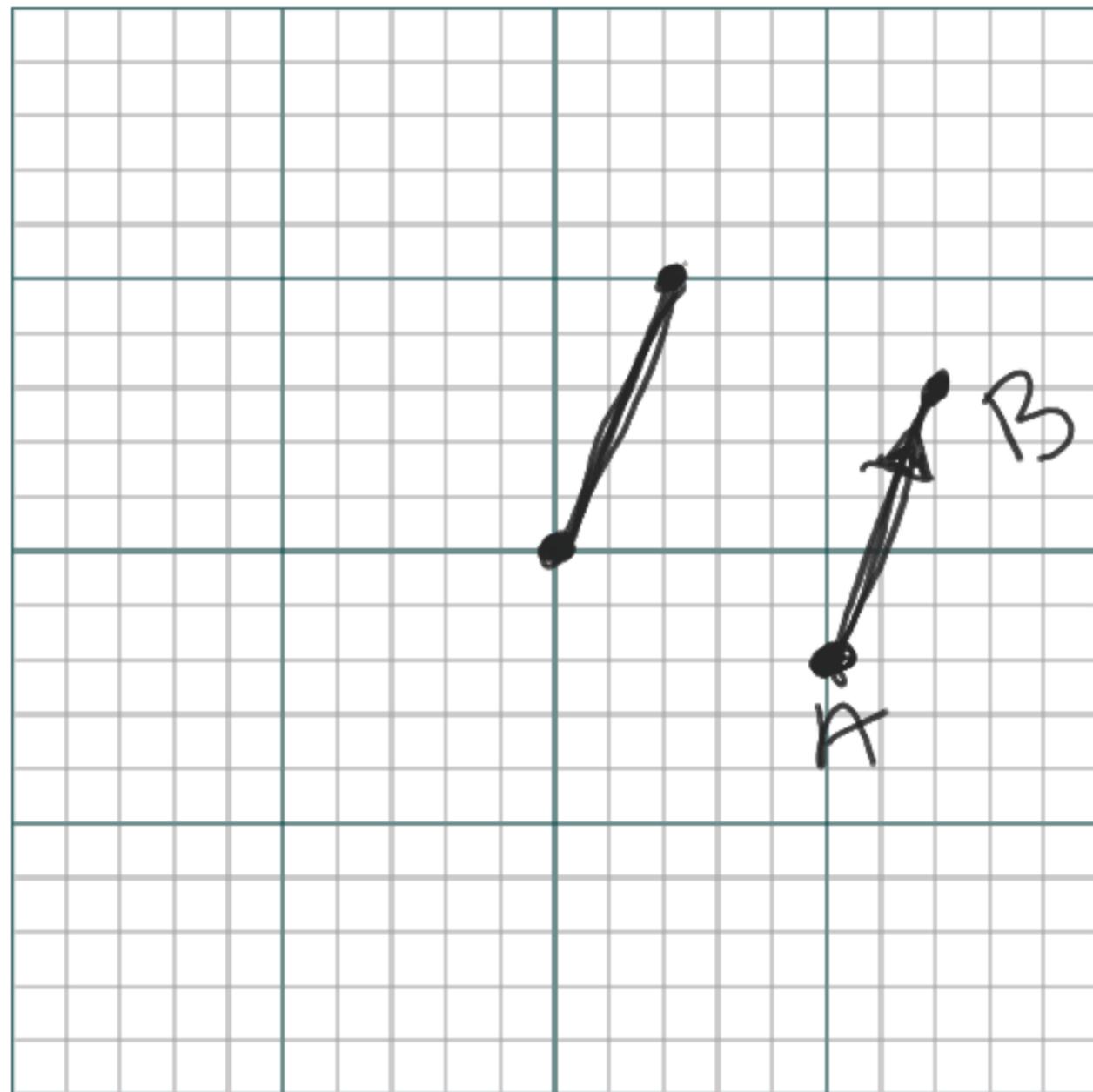
⟨7 - 5, 3 - -2⟩

⟨2, 5⟩

The **magnitude** or **length** of the vector **v** is the length of any of its representations and is denoted by the symbol $|\mathbf{v}|$ or $\|\mathbf{v}\|$. By using the distance formula to compute the length of a segment OP, we obtain the following formulas.

The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$
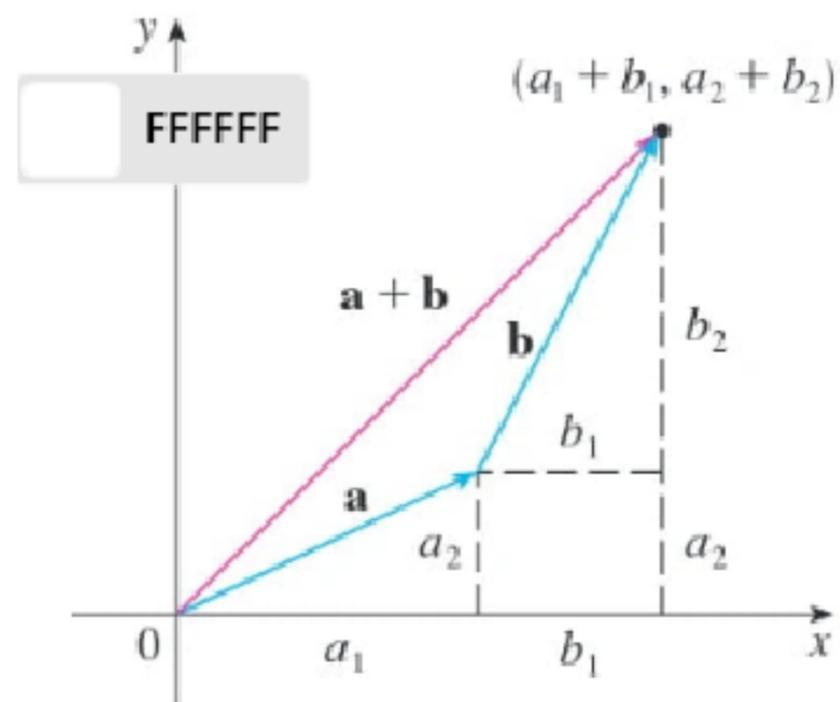
$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

EXAMPLE 4 If $\mathbf{a} = \langle 4, 0, 3 \rangle$ and $\mathbf{b} = \langle -2, 1, 5 \rangle$, find $|\mathbf{a}|$ and the vectors $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $3\mathbf{b}$, and $2\mathbf{a} + 5\mathbf{b}$.

$$|a| \rightarrow \sqrt{(4)^2 + (0)^2 + (3)^2} = \sqrt{25} = 5$$

$$|b| \rightarrow \sqrt{(-2)^2 + (1)^2 + (5)^2} = \sqrt{30}$$

$$a + b \rightarrow \langle 4 + -2, 0 + 1, 3 + 5 \rangle = \langle 2, 1, 8 \rangle$$

$$a - b \rightarrow \langle 4 - -2, 0 - 1, 3 - 5 \rangle$$

$$\rightarrow \langle 6, -1, -2 \rangle$$

EXAMPLE 4 If $\mathbf{a} = \langle 4, 0, 3 \rangle$ and $\mathbf{b} = \langle -2, 1, 5 \rangle$, find $|\mathbf{a}|$ and the vectors $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $3\mathbf{b}$, and $2\mathbf{a} + 5\mathbf{b}$.

$$3b \rightarrow 3\langle -2, 1, 5 \rangle \rightarrow \langle -6, 3, 15 \rangle$$

$$2a + 5b \rightarrow 2\langle 4, 0, 3 \rangle + 5\langle -2, 1, 5 \rangle$$

$$\langle 8, 0, 6 \rangle + \langle -10, 5, 25 \rangle$$

$$\boxed{\langle -2, 5, 31 \rangle}$$

Properties of Vectors If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$

4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

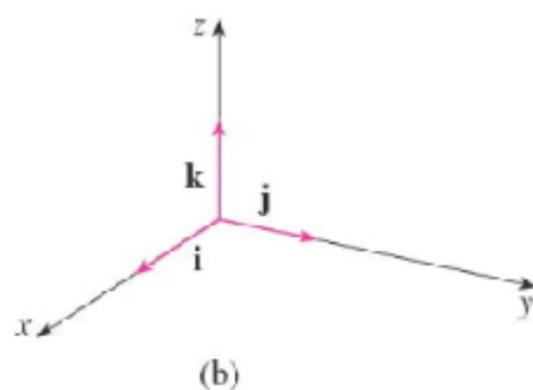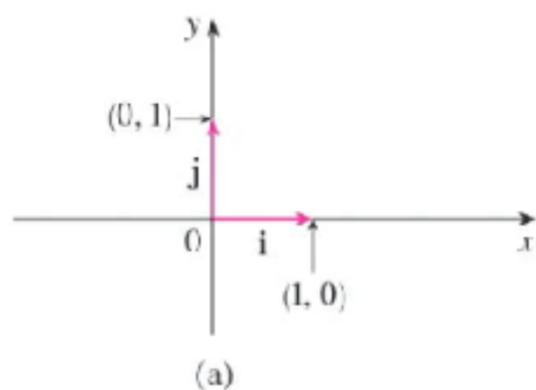5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$

6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$

7. $(cd)\mathbf{a} = c(d\mathbf{a})$

8. $1\mathbf{a} = \mathbf{a}$

$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$

These vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are called the **standard basis vectors**. They have length 1 and point in the directions of the positive x-, y-, and z-axes. Similarly, in two dimensions we define $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$. (See Figure 17.)

(a)

(b)

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then we can write

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$

$$= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle$$

2
$$\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$$

Thus any vector in V_3 can be expressed in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. For instance,

$$\langle 1, -2, 6 \rangle = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

Similarly, in two dimensions, we can write

3
$$\mathbf{a} = \langle a_1, a_2 \rangle = a_1 \mathbf{i} + a_2 \mathbf{j}$$

EXAMPLE 5 If $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 7\mathbf{k}$, express the vector $2\mathbf{a} + 3\mathbf{b}$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

$$2a + 3b \rightarrow 2(i + 2j - 3k) + 3(4i + 7k)$$

$$\rightarrow 2i + 4j - 6k + 12i + 21k$$

$$\rightarrow \boxed{14i + 4j + 15k}$$

A **unit vector** is a vector whose length is 1. For instance, **i**, **j**, and **k** are all unit vectors. In general, if $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as **a** is

$$\boxed{4} \qquad \mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

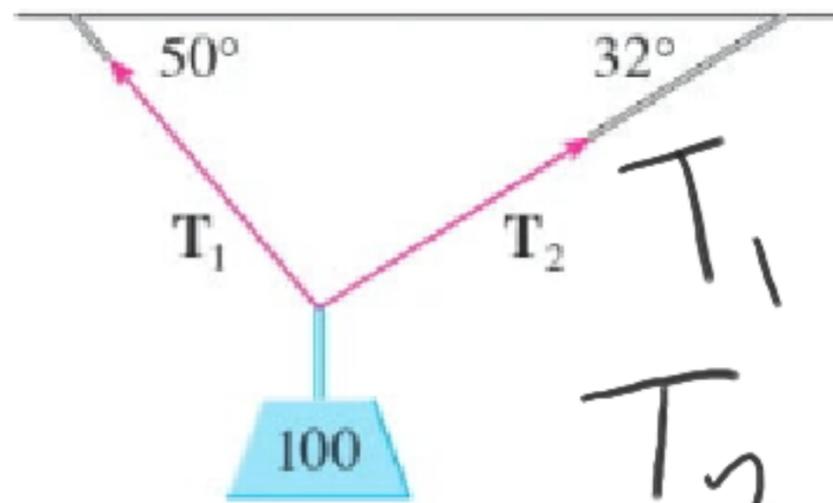**EXAMPLE 6** Find the unit vector in the direction of the vector $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

$$|2i - j - 2k| = \sqrt{(2)^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

$$\frac{1}{3}(2i - j - 2k) = \frac{2}{3}i - \frac{1}{3}j - \frac{2}{3}k$$

$$\text{unit vector}$$

EXAMPLE 7 A 100-lb weight hangs from two wires as shown in Figure 19. Find the tensions (forces) $\mathbf{T}_1$ and $\mathbf{T}_2$ in both wires and the magnitudes of the tensions.

$$T_1 = |T_1|(-\cos 50)i + |T_1|(\sin 50)j$$

$$T_2 = |T_2|(\cos 32)i + |T_2|(\sin 32)j$$
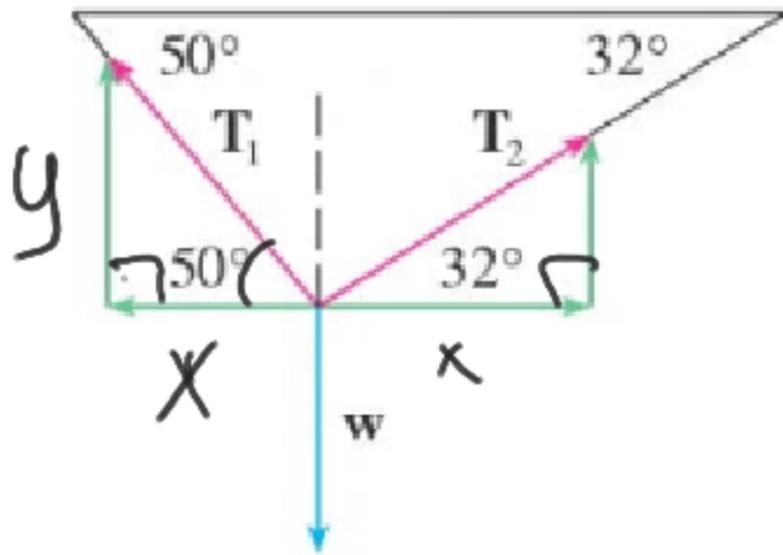
FIGURE 20

$$\cos 50 = \frac{x}{T_1} \qquad \sin 50 = \frac{y}{T_1}$$

$$T_1(\cos 50) = x \qquad T_1(\sin 50)$$

$$T_1 = |T_1| \cos 50\, i + |T_1| \sin 50\, j$$
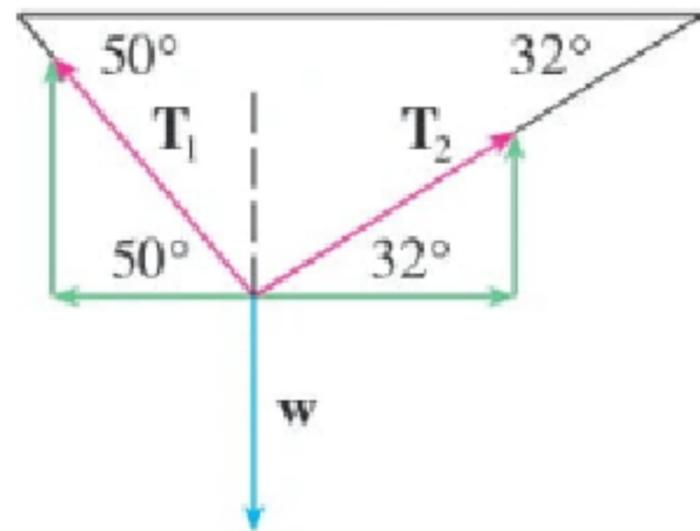
$$T_2 = |T_2| \cos 32\, i + |T_2| \sin 32\, j$$

FIGURE 20

$$T_1 + T_2 = -W = 100\, j \qquad W = -100\, j$$

$$-|T_1| \cos 50\, i + |T_1| \sin 50\, j + |T_2| \cos 32\, i + |T_2| \sin 32\, j = 100\, j$$

$$-|T_1| \cos 50\, i + |T_2| \cos 32\, i = 0\, i$$

$$|T_1| \sin 50\, j + |T_2| \sin 32\, j = 100\, j$$

$$-|T_1|\cos 50i + |T_2|\cos 32i = 0i \qquad \rightarrow \qquad |T_2| = \frac{|T_1|\cos 50}{\cos 32}$$

$$|T_1|\sin 50 j + |T_2|\sin 32 j = 100 j$$

$$|T_1|\sin 50 + \frac{|T_1|\cos 50}{\cos 32}(\sin 32) = 100$$

$$|T_1|\left(\sin 50 + \frac{\cos 50 \sin 32}{\cos 32}\right) = 100$$

$$|T_1| = \frac{100}{\sin 50 + \cos 50 \tan 32} = 85.64$$

$$|T_2| = \frac{|T_1|\cos 50}{\cos 32} = \frac{85.64 \cos 50}{\cos 32} = 64.91$$

$$T_1 = |T_1| \cos 50 \, i + |T_1| \sin 50 \, j \qquad |T_1| = \frac{100}{\sin 50 + \cos 50 \tan 32} = 85.64$$

$$T_2 = |T_2| \cos 32 \, i + |T_2| \sin 32 \, j \qquad |T_2| = \frac{|T_1| \cos 50}{\cos 32} = \frac{85.64 \cos 50}{\cos 32} = 64.91$$

$$T_1 = 85.64 \cos 50 \, i + 85.64 \sin 50 \, j$$

$$\boxed{T_1 = -55.05 \, i + 65.60 \, j}$$

$$T_2 = 64.91 \cos 32 \, i + 64.91 \sin 32 \, j$$

$$\boxed{T_2 = 55.05 \, i + 34.40 \, j}$$