2.3 Calculating Limits Using the Limit Laws

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$
4. $\lim_{x\to a} [f(x)\, g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

6. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer

7. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer

[If n is even, we assume that $\lim_{x\to a} f(x) > 0$.]

8. $\lim_{x\to a} c = c$
9. $\lim_{x\to a} x = a$

EXAMPLE 2 Evaluate the following limits and justify each step.

(a) $\lim_{x \to 5} (2x^2 - 3x + 4)$ (b) $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$

$$\lim_{x \to 5} (2x^2) - \lim_{x \to 5} (3x) + \lim_{x \to 5} (4)$$

$$\lim_{x \to 5} (2) \cdot \left(\lim_{x \to 5} (x)\right)^2$$

$$\lim_{x \to 5} (2x^2 - 3x + 4) \to 2(5)^2 - 3(5) + 4 \to 39$$

EXAMPLE 2 Evaluate the following limits and justify each step.

(a) $\lim_{x \to 5} (2x^2 - 3x + 4)$ (b) $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$

b) $$\frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = \frac{-8 + 8 - 1}{5 + 6} = \frac{-1}{11}$$

EXAMPLE 3 Find $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

$$\frac{x^2 - 1}{x - 1} \rightarrow \frac{(x+1)(x-1)}{x-1}$$

$$\lim_{x \to 1} (x+1) = 2$$

EXAMPLE 4 Find $\lim_{x \to 1} g(x)$ where

$$\lim_{x \to 1} \quad g(x) = \begin{cases} x + 1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases} \rightarrow 2$$

$$\lim_{x \to 1^-} x + 1 \rightarrow 2$$

$$\lim_{x \to 1^+} x + 1 \rightarrow 2$$

EXAMPLE 5 Evaluate $\lim_{h \to 0} \frac{(3+h)^2 - 9}{h}$. $\to 6$

$$\frac{(3+h)^2 - 9}{h} = \frac{9 + 6h + h^2 - 9}{h}$$

$$\frac{h(6+h)}{h} = 6 + h$$

EXAMPLE 6 Find $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

$$\lim_{t\to 0} \frac{\left(\sqrt{t^2+9}-3\right)}{t^2} \cdot \frac{\left(\sqrt{t^2+9}+3\right)}{\left(\sqrt{t^2+9}+3\right)}$$

$$\frac{(t^2+9)-9}{t^2\left(\sqrt{t^2+9}+3\right)} = \frac{1}{\sqrt{t^2+9}+3} = \frac{1}{6}$$

EXAMPLE 7 Show that $\lim_{x \to 0} |x| = 0$.

$$f(x) = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

$$\lim_{x \to 0^-} -x \Rightarrow 0$$

$$\lim_{x \to 0^+} x \Rightarrow 0$$

EXAMPLE 8 Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

$$f(x) = \begin{cases} \frac{-x}{x} = -1 & x < 0 \\ \frac{x}{x} = 1 & x \geq 0 \end{cases}$$

$$\lim_{x\to 0^-} f(x) = -1 \qquad \lim_{x\to 0^+} f(x) = 1$$

$$\lim_{x \to -5} \frac{2x^2 + 9x - 5}{x^2 - 25} = \frac{(2x-1)(x+5)}{(x-5)(x+5)}$$

$$\frac{2x-1}{x-5} = \frac{-11}{-10} = \boxed{\frac{11}{10}}$$

$2x^2 + 9x - 5$

$-10 \quad 9$

$10 \quad -1$

$2x^2 + 10x \mid -1x - 5$

$2x(x+5) \quad -1(x+5)$

$(2x-1)(x+5)$

$$\lim_{h\to 0} \frac{(-2+h)^{-1} + 2^{-1}}{h}$$

$$\frac{1}{-2+h} + \frac{1}{2}$$

$$\frac{2 + (-2+h)}{2(-2+h)}$$

$$\frac{h}{-4+2h}$$

$$\frac{\frac{h}{-4+2h}}{h}$$

$$\frac{h}{-4+2h} \cdot \frac{1}{h} = \frac{1}{-4+2h} = -\frac{1}{4}$$

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$$\lim_{x\to -4} \frac{|x+4|}{2x+8}$$ DNE

$$|x+4| \rightarrow \begin{cases} -(x+4) & x < -4 \\ x+4 & x \geq -4 \end{cases}$$

$$f(x) = \begin{cases} \dfrac{-(x+4)}{2x+8} & x < -4 \\ \dfrac{x+4}{2x+8} & x \geq -4 \end{cases}$$

$$\frac{-(x+4)}{2(x+4)} = \frac{-1}{2}$$

$$\frac{x+4}{2(x+4)} = \frac{1}{2}$$

$$\lim_{x\to -4^-} f(x) = \frac{-1}{2} \qquad \lim_{x\to -4^+} f(x) = \frac{1}{2}$$

1 Definition A function f is **continuous at a number** a if

$$\lim_{x \to a} f(x) = f(a)$$

Notice that Definition 1 implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)

2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x) = f(a)$

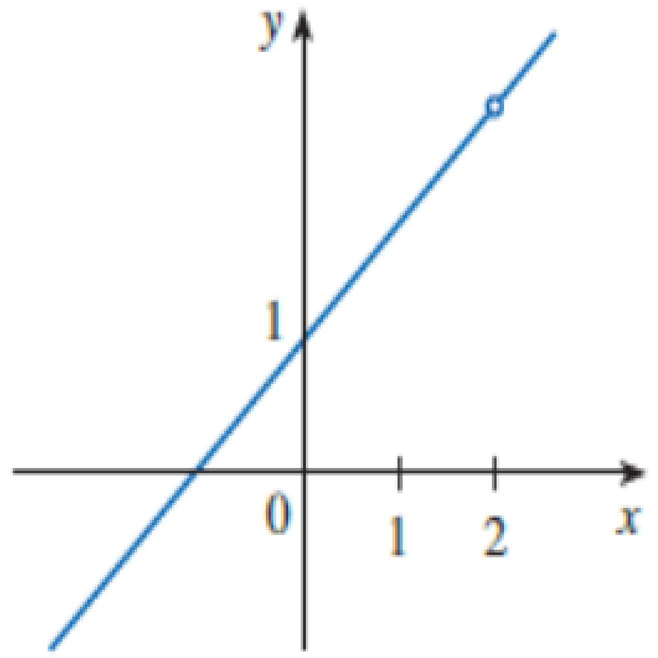
EXAMPLE 2 Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

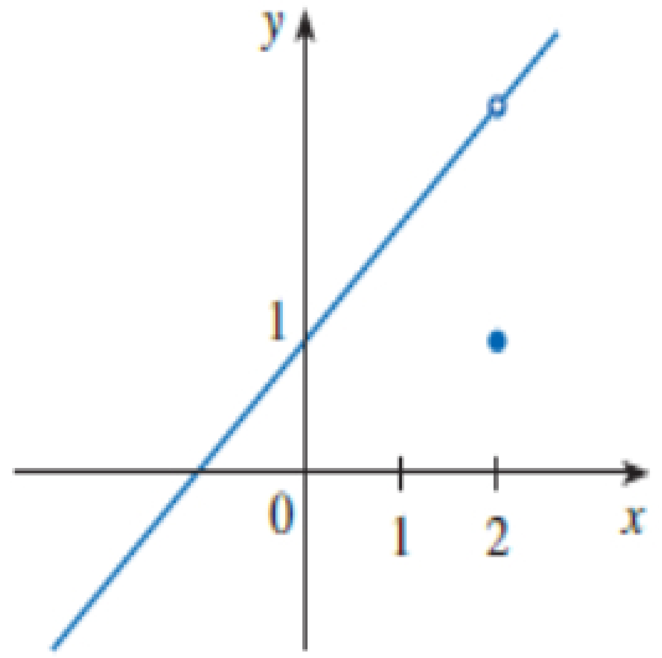
(b) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

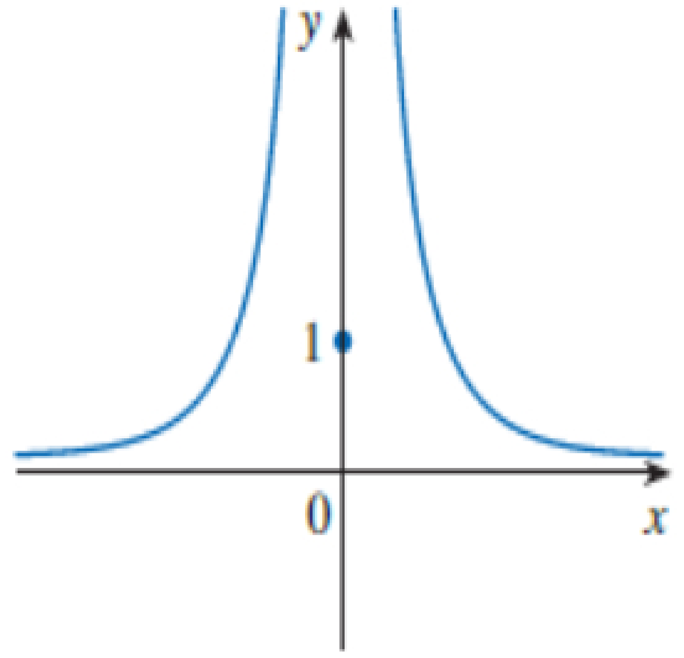
(d) $f(x) = [\![x]\!]$



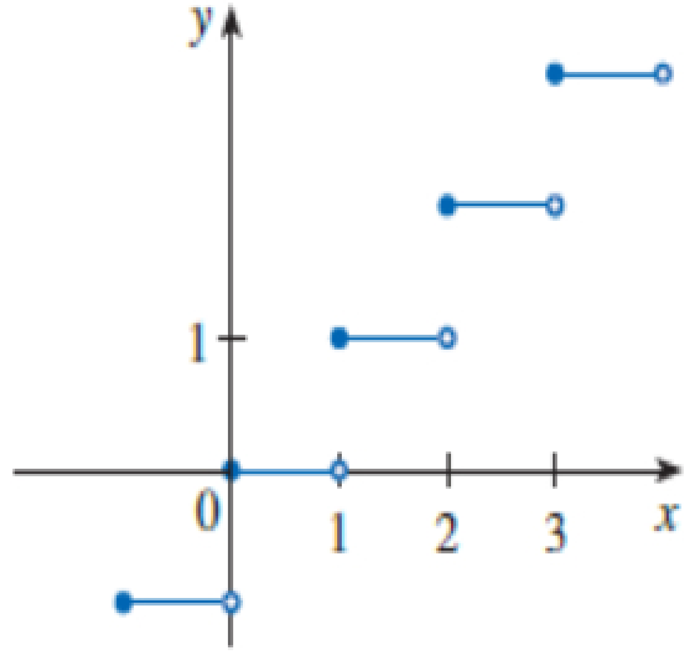
(a) A removable discontinuity



(b) A removable discontinuity



(c) An infinite discontinuity



(d) Jump discontinuities

EXAMPLE 4 Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

$$\lim_{x \to a} f(x) = f(a)$$

$$= 1 - \sqrt{1 - a^2}$$

$$\lim_{x \to -1^+} f(x) = 1 - \sqrt{1 - (-1)^2} = 1 - \sqrt{2}$$

$$\lim_{x \to 1^-} f(x) = 1 - \sqrt{1 - (1)^2} = 1 - \sqrt{2}$$

4 Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a) \neq 0$

7 Theorem The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

EXAMPLE 7 Evaluate $\lim_{x\to\pi} \frac{\sin x}{2 + \cos x}$.

$$\frac{\sin(\pi)}{2+\cos(\pi)} = \frac{0}{2+(-1)} = \frac{0}{1} = 0$$

EXAMPLE 8 Evaluate $\lim_{x\to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right)$. $\to \arcsin\left(\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}\right)$

8 Theorem If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

EXAMPLE 8 Evaluate $\lim_{x\to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right)$.

$$\to \arcsin\left(\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}\right)$$

$$\frac{(1-\sqrt{x})}{1-x}\frac{(1+\sqrt{x})}{(1+\sqrt{x})} = \frac{(1-x)}{(1-x)(1+\sqrt{x})}$$

$$\lim_{x\to 1} \frac{1}{1+\sqrt{x}} = \frac{1}{2} \to \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

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