Evaluate the integral.

$$\int_1^4 \sqrt{y}\,\ln(y)\,dy$$

$$\frac{32\ln(2)}{3} - \frac{28}{9}$$

Solution or Explanation

Let $u = \ln(y)$, $dv = \sqrt{y}\,dy$ $\Rightarrow$ $du = \frac{1}{y}\,dy$, $v = \frac{2}{3}y^{3/2}$. Then

$$\int_1^4 \sqrt{y}\,\ln(y)\,dy = \left[\frac{2}{3}y^{3/2}\ln(y)\right]_1^4 - \int_1^4 \frac{2}{3}y^{1/2}\,dy = \frac{2}{3}\cdot 8\ln(4) - 0 - \left[\frac{4}{9}y^{3/2}\right]_1^4$$

$$= \frac{16}{3}(2\ln(2)) - \left(\frac{4}{9}\cdot 8 - \frac{4}{9}\right) = \frac{32}{3}\ln(2) - \frac{28}{9}.$$

Evaluate the integral. (Use C for the constant of integration.)

$$\int \sin^5(t)\cos^4(t)\,dt$$

$$C - \frac{1}{9}\cos^9(t) + \frac{2\cos^7(t)}{7} - \frac{\cos^5(t)}{5}$$

Solution or Explanation

$$\int \sin^5(t)\cos^4(t)\,dt = \int \sin^4(t)\cos^4(t)\sin(t)\,dt = \int (\sin^2(t))^2\cos^4(t)\sin(t)\,dt$$

$$= \int (1-\cos^2(t))^2\cos^4(t)\sin(t)\,dt = \int (1-u^2)^2u^4(-du) \quad [u = \cos(t),\ du = -\sin(t)\,dt]$$

$$= \int (-u^4 + 2u^6 - u^8)\,du = -\frac{1}{5}u^5 + \frac{2}{7}u^7 - \frac{1}{9}u^9 + C = -\frac{1}{5}\cos^5(t) + \frac{2}{7}\cos^7(t) - \frac{1}{9}\cos^9(t) + C$$

Evaluate the integral. (Use C for the constant of integration.)

$$\int 3e^6 \, dx$$

Since $6e^5$ is a constant, $\int 6e^5 dx = 6e^5 x + C$.

Evaluate the integral. (Remember to use absolute values where appropriate. Use C for the constant of integration.)

$$\int 5x^3(x-1)^{-4}\,dx$$

$$C + 5\ln(|x-1|) - 15(x-1)^{-1} - \frac{15}{2}(x-1)^{-2} - \frac{5}{3}(x-1)^{-3}$$

Let $u = x - 1$, so that $du = dx$. Then

$$\begin{aligned}\int 5x^3(x-1)^{-4}\,dx &= \int 5(u+1)^3u^{-4}\,du = \int 5(u^3+3u^2+3u+1)u^{-4}\,du \\ &= \int 5(u^{-1}+3u^{-2}+3u^{-3}+u^{-4})\,du \\ &= 5\ln|u| - 15u^{-1} - \tfrac{15}{2}u^{-2} - \tfrac{5}{3}u^{-3} + C \\ &= 5\ln|x-1| - 15(x-1)^{-1} - \tfrac{15}{2}(x-1)^{-2} - \tfrac{5}{3}(x-1)^{-3} + C\end{aligned}$$

Evaluate the integral. (Use C for the constant of integration.)

$$\int (4x + 3\sin(x))^2\,dx$$

$$C + \frac{16x^3}{3} + \frac{9x}{2} + 24\sin(x) - 24x\cos(x) - \frac{9}{2}\sin(x)\cos(x)$$

$$\begin{aligned}\int (4x + 3\sin x)^2\,dx &= \int \left(16x^2 + 24x\sin x + 9\sin^2 x\right)dx \\ &= \tfrac{16}{3}x^3 + 24(\sin x - x\cos x) + \tfrac{9}{2}(x - \sin x\cos x) + C \\ &= \tfrac{16}{3}x^3 + \tfrac{9}{2}x + 24\sin x - \tfrac{9}{2}\sin x\cos x - 24x\cos x + C\end{aligned}$$

Evaluate the integral. (Use C for the constant of integration.)

$$\int \frac{11x^3}{\sqrt{8 + x^2}}\, dx$$

$$C + \frac{11}{3}\left(x^2+8\right)^{3/2} - 88\sqrt{x^2+8}$$

Let $u = 8 + x^2$, so that $du = 2x dx$. Then

$$\int \frac{11x^3}{\sqrt{8+x^2}} dx = \int \frac{11x^2}{\sqrt{8+x^2}}(x dx) = 11\int \frac{u-8}{u^{1/2}}\left(\tfrac{1}{2}du\right)$$

$$= \tfrac{11}{2}\int (u^{1/2} - 8u^{-1/2})du = \frac{11}{2}\left(\frac{2}{3}u^{3/2} - 16u^{1/2}\right) + C$$

$$= \tfrac{11}{3}(8+x^2)^{3/2} - 88(8+x^2)^{1/2} + C \quad \left[\text{or } \frac{11}{3}\left(x^2-16\right)\sqrt{8+x^2} + C\right]$$

Evaluate the integral. (Remember to use absolute values where appropriate. Use C for the constant of integration.)

$$\int \frac{5x^2 - 6}{x^2 - 4x - 12}\,dx$$

$$C + 5x + \frac{87}{4}\ln(|x-6|) - \frac{7}{4}\ln(|x+2|)$$

$$\frac{5x^2-6}{x^2-4x-12} = 5 + \frac{20x+54}{(x-6)(x+2)} = 5 + \frac{A}{x-6} + \frac{B}{x+2} \quad \Rightarrow$$

$20x + 54 = A(x+2) + B(x-6)$. Setting $x = 6$ gives $174 = 8A$, so $A = \frac{87}{4}$.

Setting $x = -2$ gives $14 = -8B$, so $B = -\frac{7}{4}$. Now $\int \frac{5x^2-6}{x^2-4x-12}\,dx =$
$\int \left(5 + \frac{87/4}{x-6} - \frac{7/4}{x+2}\right) dx$

$= 5x + \frac{87}{4}\ln|x-6| - \frac{7}{4}\ln|x+2| + C.$

Evaluate the integral. (Remember to use absolute values where appropriate.)

$$\int_0^2 \frac{6t}{(t-3)^2}\,dt$$

$12 - 6\ln(3)$

$$\int_0^2 \frac{6t}{(t-3)^2}\,dt = \int_{-3}^{-1} \frac{6(u+3)}{u^2}\,du \quad \begin{bmatrix} u & = t-3, \\ du & = dt \end{bmatrix} = \int_{-3}^{-1} \left(\frac{6}{u} + \frac{18}{u^2}\right) du$$

$$= \left[6\ln|u| - \frac{18}{u}\right]_{-3}^{-1} = (6\ln 1 + 18) - (6\ln 3 + 6)$$

$$= 12 - 6\ln 3 \quad \text{or} \quad 12 - \ln 3^6$$

Evaluate the integral. (Remember to use absolute values where appropriate. Use C for the constant of integration.)

$$\int 5e^x\sqrt{6 + e^x}\,dx$$

$C + \frac{10}{3}(6+e^x)^{3/2}$

Let $u = 6 + e^x$, so that $du = e^x\,dx$.

Then $\int 5e^x\sqrt{6+e^x}\,dx = \int 5u^{1/2}\,du = \frac{10}{3}u^{3/2} + C = \frac{10}{3}(6+e^x)^{3/2} + C$.

Or: Let $u = \sqrt{6+e^x}$, so that $u^2 = 1 + e^x$ and $2u\,du = e^x\,dx$. Then $\int 5e^x\sqrt{6+e^x}\,dx = \int 5u \cdot 2u\ du = \int 10u^2\,du = \frac{10}{3}u^3 + C$
$= \frac{10}{3}(6+e^x)^{3/2} + C$.

EXAMPLE 5 $\int \sqrt{\dfrac{7-x}{7+x}}\,dx.$

Although the rationalizing substitution

$$u = \sqrt{\frac{7-x}{7+x}}$$

works here, it leads to a very complicated rational function. An easier method is to do some algebraic manipulation. Multiplying numerator and denominator by $\sqrt{7-x}$, we have

$$\int \frac{7-x}{\sqrt{49-x^2}}\,dx = \int \left(\boxed{} \;\boxed{\frac{7}{\sqrt{49-x^2}}} \right) dx - \int \frac{x}{\sqrt{49-x^2}}\,dx$$

$$= \boxed{} \;\boxed{\sqrt{49-x^2} + 7\sin^{-1}\left(\frac{x}{7}\right)} + C.$$