1)

Find the exact length of the curve.

$y = 4 + 8x^{3/2}, \quad 0 \le x \le 1$

$$\frac{1}{216}\left(145\sqrt{145} - 1\right)$$

$$y = 4 + 8x^{3/2} \quad \Rightarrow \quad dy/dx = 12x^{1/2} \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + 144x.$$

So $L = \int_0^1 \sqrt{1 + 144x}\, dx = \int_1^{145} u^{1/2}\left(\frac{1}{144}\, du\right) \quad \left[\begin{matrix} u = 1 + 144x, \\ du = 144\, dx \end{matrix}\right] \quad = \frac{1}{144}\cdot\frac{2}{3}\left[u^{3/2}\right]_1^{145}$

$= \frac{1}{216}\left(145\sqrt{145} - 1\right)$

2)

Find the exact length of the curve.

$y = \frac{1}{4}x^2 - \frac{1}{2}\ln(x), \quad 1 \le x \le 4$

$$\frac{15}{4} + \frac{\ln(4)}{2}$$

Solution or Explanation

$y = \frac{1}{4}x^2 - \frac{1}{2}\ln(x) \quad \Rightarrow \quad y' = \frac{1}{2}x - \frac{1}{2x} \quad \Rightarrow \quad 1 + (y')^2 = 1 + \left(\frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2}\right) = \frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{1}{2}x + \frac{1}{2x}\right)^2.$ So

$$L = \int_1^4 \sqrt{1 + (y')^2}\, dx = \int_1^4 \left|\frac{1}{2}x + \frac{1}{2x}\right| dx = \int_1^4 \left(\frac{1}{2}x + \frac{1}{2x}\right) dx$$

$$= \left[\frac{1}{4}x^2 + \frac{1}{2}\ln(|x|)\right]_1^4 = \left(4 + \frac{1}{2}\ln(4)\right) - \left(\frac{1}{4} + 0\right) = \frac{15}{4} + \frac{1}{2}\ln(4).$$

3)

Find the exact area of the surface obtained by rotating the curve about the x-axis.

$$y = \sqrt{1 + e^x}, \quad 0 \le x \le 9$$

$(17 + e^9)\,\pi$

Solution or Explanation

$$y = \sqrt{1+e^x} \;\Rightarrow\; y' = \tfrac{1}{2}(1+e^x)^{-1/2}(e^x) = \frac{e^x}{2\sqrt{1+e^x}}$$

$$\Rightarrow\; \sqrt{1+(y')^2} = \sqrt{1 + \frac{e^{2x}}{4(1+e^x)}} = \sqrt{\frac{4+4e^x+e^{2x}}{4(1+e^x)}} = \sqrt{\frac{(e^x+2)^2}{4(1+e^x)}} = \frac{e^x+2}{2\sqrt{1+e^x}}$$

So

$$S = \int_0^9 2\pi y\sqrt{1+(y')^2}\,dx = 2\pi\int_0^9 \sqrt{1+e^x}\,\frac{e^x+2}{2\sqrt{1+e^x}}\,dx = \pi\int_0^9 (e^x+2)\,dx$$

$$= \pi\Big[e^x + 2x\Big]_0^9 = \pi\Big[(e^9+18) - (1+0)\Big] = \pi(e^9+17).$$

4)

The given curve is rotated about the y-axis. Find the area of the resulting surface.

$$y = \frac{1}{4}x^2 - \frac{1}{2}\ln(x), \quad 2 \le x \le 3$$

$\frac{22\pi}{3}$

$$y = \tfrac{1}{4}x^2 - \tfrac{1}{2}\ln x \;\Rightarrow\; \frac{dy}{dx} = \frac{x}{2} - \frac{1}{2x} \;\Rightarrow\; 1 + \left(\frac{dy}{dx}\right)^2$$

$$= 1 + \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2} = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{x}{2} + \frac{1}{2x}\right)^2. \text{ So}$$

$$S = \int_2^3 2\pi x\sqrt{\left(\frac{x}{2}+\frac{1}{2x}\right)^2}\,dx = 2\pi\int_2^3 x\left(\frac{x}{2}+\frac{1}{2x}\right)dx = \pi\int_2^3 (x^2+1)\,dx = \pi\left[\tfrac{1}{3}x^3 + x\right]_2^3$$

$$= \pi\left[\left(\tfrac{27}{3}+3\right) - \left(\tfrac{8}{3}+2\right)\right] = \tfrac{22}{3}\pi$$

5)

This question has several parts that must be completed sequentially. If you skip a part of the question, you will not recei

Let $f(x) = \frac{c}{1 + x^2}$.

Exercise (a)

For what value of c is f a probability density function?

Step 1

In order to be a probability density function, $f(x)$ must be non-negative and it must be true that

$\int_{-\infty}^{\infty} f(x)\,dx =$ [ ] 1 .

Step 2

As long as $c \geq 0$, then we will have $f(x) = \frac{c}{1 + x^2} \geq 0$.

We can find c by solving $\int_{-\infty}^{\infty} \frac{c}{1 + x^2}\,dx =$ [ ] 1 .

Step 3

For positive x, we have the following.

$$\int_0^{\infty} \frac{c}{1 + x^2}\,dx = \lim_{b \to \infty} \int_0^b \frac{c}{1 + x^2}\,dx$$

$$= c \lim_{b \to \infty} \Big[\text{[ ]} \; \tan^{-1}(x) \Big]_0^b$$

$$= \text{[ ]} \; c\frac{\pi}{2}$$

Step 4

By symmetry, $\int_{\infty}^{0} \frac{c}{1 + x^2}\,dx = c\,\frac{\pi}{2}$.

Therefore $\int_{-\infty}^{\infty} \frac{c}{1 + x^2}\,dx =$ [ ] $c\pi$, and $c =$ [ ] $\frac{1}{\pi}$.

Exercise (b)

For that value of c, find $P(-6 < X < 6)$.

Step 1

$$P(-6 < X < 6) = \int_{\text{[ ] } -6}^{\text{[ ] } 6} \frac{1/\pi}{1 + x^2}\,dx$$

Step 2

$$P(-6 < X < 6) = \int_{-6}^{6} \frac{1/\pi}{1 + x^2}\,dx$$

$$= \frac{2}{\pi} \int_0^6 \frac{1}{1 + x^2}\,dx$$

$$= \frac{2}{\pi} \Big[\text{[ ]} \; \tan^{-1}(x) \Big]_0^6$$

= [ ] 0.895 (Round your answer to three decimal places.)

6)

An online retailer has determined that the average time for credit card transactions to be electronically approved is 1.5 seconds. (Round your answers to three decimal places.)

(a) Use an exponential density function to find the probability that a customer waits less than a second for credit card approval.

[ ] 0.487

(b) Find the probability that a customer waits more than 3 seconds.

[ ] 0.135

(c) What is the minimum approval time for the slowest 5% of transactions?

[ ] 4.494 sec

Solution or Explanation

(a) An exponential density function with $\mu = 1.5$ is $f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1.5}e^{-t/1.5} & \text{if } t \geq 0 \end{cases}$.

The probability that a customer waits less than a second is

$$P(X < 1) = \int_0^1 f(t)\,dt = \int_0^1 \frac{1}{1.5}e^{-t/1.5}\,dt = \left[-e^{-t/1.5}\right]_0^1 = -e^{-1/1.5} + 1 \approx 0.487.$$

(b) The probability that a customer waits more than 3 seconds is

$$P(X > 3) = \int_3^\infty f(t)\,dt = \lim_{s \to \infty} \int_3^s f(t)\,dt = \lim_{s \to \infty} \left[-e^{-t/1.5}\right]_3^s = \lim_{s \to \infty} \left(-e^{-s/1.5} + e^{-3/1.5}\right) = e^{-3/1.5} \approx 0.135.$$

Or: Calculate $1 - \int_0^3 f(t)\,dt$.

(c) We want to find b such that $P(X > b) = 0.05$. From part (b), $P(X > b) = e^{-b/1.5}$. Solving $e^{-b/1.5} = 0.05$ gives us

$$-\frac{b}{1.5} = \ln(0.05) \quad \Rightarrow \quad b = -1.5\ln(0.05) \approx 4.494 \text{ seconds.}$$

Or: Solve $\int_0^b f(t)\,dt = 0.95$ for b.

7)

The speeds of vehicles on a highway with speed limit 110 km/h are normally distributed with mean 120 km/h and standard deviation 5 km/h. (Round your answers to two decimal places.)

(a) What is the probability that a randomly chosen vehicle is traveling at a legal speed?

[ ] 2.28 %

(b) If police are instructed to ticket motorists driving 130 km/h or more, what percentage of motorist are targeted?

[ ] 2.28 %

(a) $P(0 \le X \le 110) = \int_0^{110} \frac{1}{5\sqrt{2\pi}} \exp\left(-\frac{(x-120)^2}{2 \cdot 5^2}\right) dx \approx 0.0228$ (using a calculator or computer to estimate the integral), so there is about a 2.28% chance that a randomly chosen vehicle is traveling at a legal speed.

(b) $P(X \ge 130) = \int_{130}^{\infty} \frac{1}{5\sqrt{2\pi}} \exp\left(-\frac{(x-120)^2}{2 \cdot 5^2}\right) dx = \int_{130}^{\infty} f(x)\,dx$. In this case, we could use a calculator or computer to estimate either $\int_{130}^{300} f(x)\,dx$ or $1 - \int_0^{130} f(x)\,dx$. Both are approximately 0.0228, so about 2.28% of the motorists are targeted.

8)

Solve the differential equation.

$$\frac{dy}{dx} = 5x\sqrt{y}$$

[ ] $y = \left(C + \frac{5x^2}{4}\right)^2$ for $y \ne 0$

Solution or Explanation

$\frac{dy}{dx} = 5x\sqrt{y} \Rightarrow \frac{dy}{\sqrt{y}} = 5x\,dx\ [y \ne 0] \Rightarrow \int y^{-1/2}\,dy = \int 5x\,dx \Rightarrow 2y^{1/2} = \frac{5}{2}x^2 + K \Rightarrow \sqrt{y} = \frac{5}{4}x^2 + \frac{1}{2}K \Rightarrow$

$y = \left(\frac{5}{4}x^2 + C\right)^2$, where $C = \frac{1}{2}K$. $y = 0$ is also a solution.

9)

Solve the differential equation.

$$\frac{du}{dt} = \frac{9 + t^4}{ut^2 + u^4t^2}$$

$\frac{u^5}{5} + \frac{u^2}{2} = C + \frac{t^3}{3} - \frac{9}{t}$

Solution or Explanation

$\frac{du}{dt} = \frac{9 + t^4}{ut^2 + u^4t^2} \Rightarrow \frac{du}{dt} = \frac{9 + t^4}{t^2(u + u^4)} \Rightarrow (u + u^4)\,du = \frac{9 + t^4}{t^2}\,dt \Rightarrow \int (u + u^4)\,du = \int (9t^{-2} + t^2)\,dt \Rightarrow$

$\frac{1}{2}u^2 + \frac{1}{5}u^5 = -\frac{9}{t} + \frac{1}{3}t^3 + C.$ We cannot solve explicitly for u.

10)

Tutorial Exercise

Find the solution of the differential equation that satisfies the given initial condition.

$$\frac{du}{dt} = \frac{2t + \sec^2(t)}{2u}, \quad u(0) = -10$$

Step 1

Separating $\frac{du}{dt} = \frac{2t + \sec^2(t)}{2u}$ gives us $2u\,du = \left(\; 2t + \sec^2(t) \;\right) dt.$

Step 2

Ignoring the constant of integration, integrating the left side gives $\int 2u\,du =$ u^2 .

Step 3

Ignoring the constant of integration, integrating the right side gives

$\int (2t + \sec^2(t))\,dt =$ $t^2 + \tan(t)$.

Step 4

We now have $u^2 = t^2 + \tan(t) + C$.

Since $u(0) = -10$, we can substitute into the equation and solve for $C =$ 100 .

Step 5

Noting that $u(0) = -10$ is negative, then when we solve for u we must get $u =$

$-\sqrt{t^2 + \tan(t) + 100}$.