

# 6

## CHAPTER

# Probability, Randomness, and Uncertainty

6.1 Introduction to Probability

6.2 Addition Rules for Probability

## Random Experiment

A **random experiment** is defined as any activity or phenomenon that meets the following conditions.

1. There is one distinct outcome for each trial of the experiment.
2. The outcome of the experiment is uncertain.
3. The set of all distinct outcomes of the experiment can be specified and is called the **sample space**, denoted by  $S$ .

**DEFINITION**

## Outcome

An **outcome** is any member of the sample space.

DEFINITION

## Event

An **event** is a set of outcomes.

DEFINITION

**Experiment 1:** Toss a coin and observe the outcome. Have we met the three conditions of a random experiment?

$$S = \{ \text{Head, Tail} \}$$

**Experiment 2:** Toss a coin three times and observe the number of heads. Have we met the three conditions of a random experiment?

**Experiment 4:** Assume we have a deck of playing cards consisting of 13 hearts, 13 clubs, 13 spades, and 13 diamonds. Draw a card from a well-shuffled deck and observe the suit of the card. Have we met the three conditions of a random experiment?

**Experiment 5:** Inspect a transistor to determine if it meets quality control standards.  
Have we met the three conditions of a random experiment?

$$S = \{ \text{pass}, \text{fail} \}$$

## Relative Frequency

If an experiment is performed  $n$  times, under identical conditions, and the event  $A$  happens  $k$  times, the **relative frequency** of  $A$  is given by the following expression.

$$\text{Relative Frequency of } A = \frac{k}{n}$$

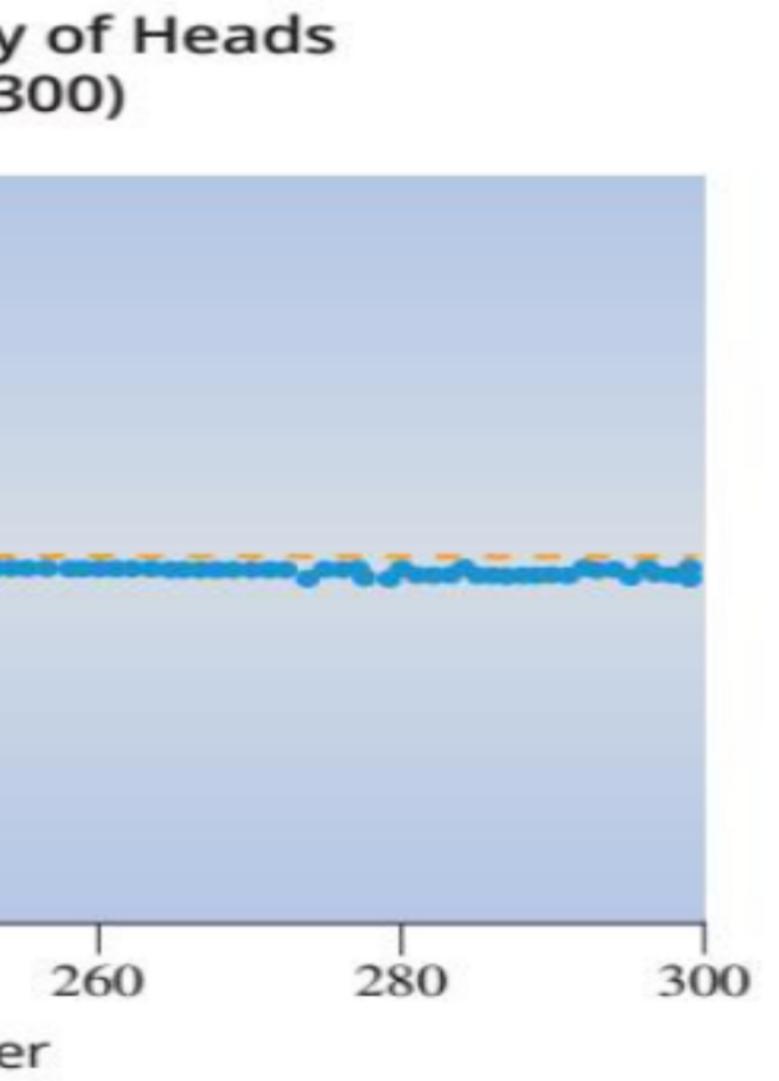
If the relative frequency converges as  $n$  increases, then the relative frequency is said to be the **probability** of  $A$ .

**DEFINITION**

Experimental

0 10 20 30 40  
Flip Number

Figure 6.1.1



Relative Frequency of Heads  
(Flips 1350 - 1450)

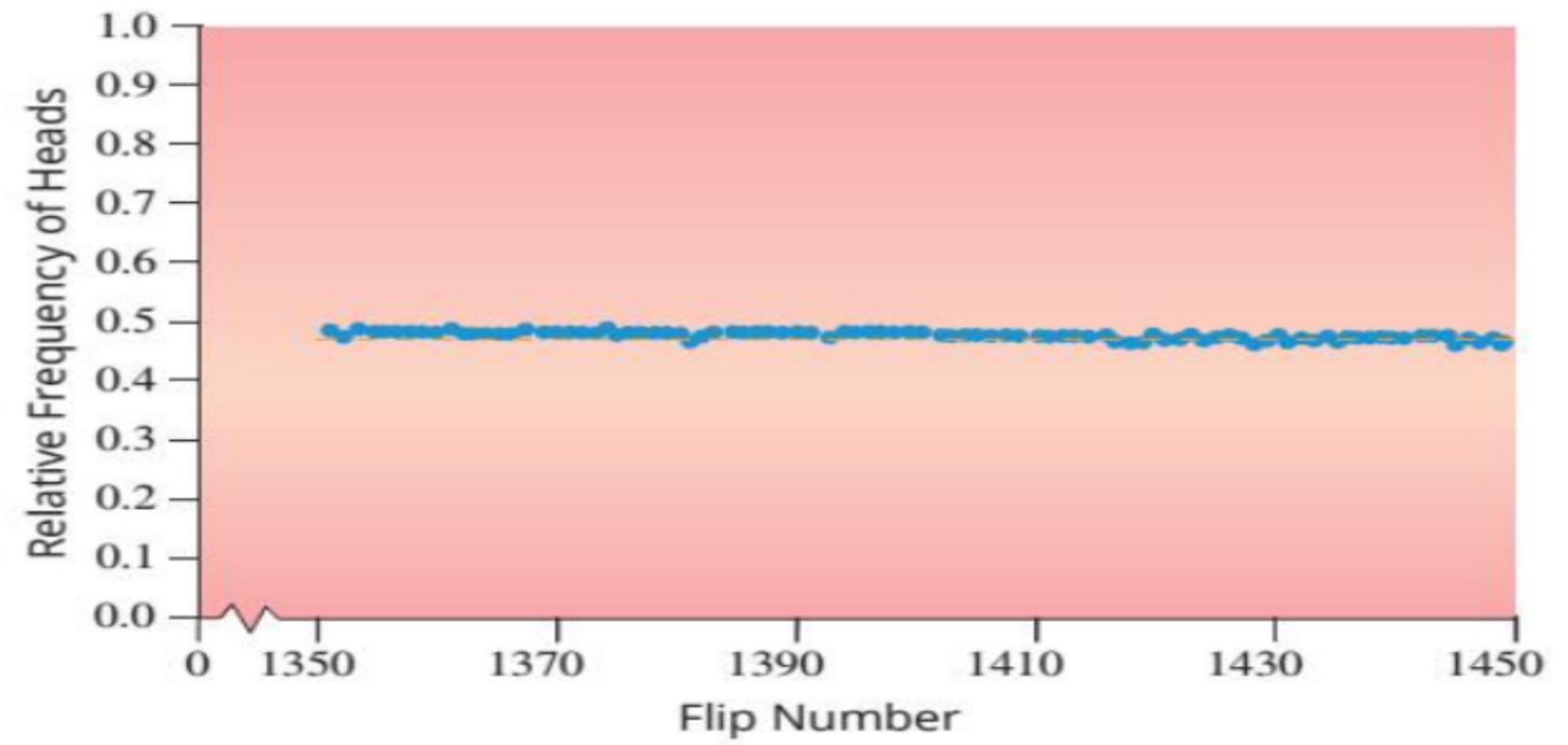


Figure 6.1.3

## Summary

**The experiment:** Toss a coin and observe which side of the coin appears on top.

**Duration of the experiment:** Toss the coin 1450 times.  $n = 1450$

**Observe the event “getting a head”:** The event was observed 718 times.  $k = 718$

**Relative frequency of the event “getting a head”:**

$$A = \frac{k}{n} = \frac{718}{1450} \approx 0.4952$$

## Classical Probability

Using the **classical approach** to probability, the probability of an event  $A$ , denoted  $P(A)$ , is given by

$$P(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes in the sample space}}.$$

**DEFINITION**

Theoretical

①

H  
T

↘ ↗  
↗ ↘

②

H  
T

↘ ↗ ↘ ↗  
↗ ↘ ↗ ↘ ↗ ↘

H  
T

↘ ↗ ↘ ↗ ↘ ↗ ↘ ↗  
↗ ↘ ↗ ↘ ↗ ↘ ↗ ↘ ↗ ↘

③

H	H	H	/
H	H	T	/
H	T	H	/
H	T	T	/
T	H	H	/
T	H	T	/
T	T	H	/
T	T	T	/

$$\frac{4}{8}$$

13. A gambler has made a weighted die. In order to decide which of the six sides is most likely to turn up, he tosses the die 33 times and notes the number of dots on the upper-most surface. The results of the experiment are shown in the following table.

Rolls of a Weighted Die										
1	2	1	3	1	4	1	5	6	3	1
3	1	5	1	2	1	3	1	2	1	2
2	1	3	5	1	2	1	2	1	4	6

$$1) \frac{14}{33}$$

$$2) \frac{7}{33}$$

$$3) \frac{5}{33}$$

$$4) \frac{2}{33}$$

$$5) \frac{3}{33}$$

$$6) \frac{2}{33}$$

- Using the relative frequency approach, what is the probability of observing each side?
- Assuming all outcomes have an equal payoff, which side do you think the gambler will bet on when the die is tossed?

## Probability Law 1

A probability of zero means the event cannot happen. (For example, the probability of observing three heads in two tosses of a coin is zero.)

**DEFINITION**

## Probability Law 2

A probability of one means the event must happen. (For example, if we toss a coin, the probability of getting either a head or tail is one.)

**DEFINITION**

## Probability Law 3

All probabilities must be between zero and one inclusively. The closer the probability is to 1, the more likely the event. The closer the probability is to 0, the less likely the event. For an event  $A$  this is expressed as follows.

$$0 \leq P(A) \leq 1$$

**DEFINITION**

## Probability Law 4

The sum of the probabilities of all outcomes must equal one. That is, if  $P(A_i)$  is the probability of event  $A_i$ , and there are  $n$  such outcomes, then  $P(A_1) + P(A_2) + \dots + P(A_n) = 1$ .

**DEFINITION**

## Compound event

A **compound event** is an event that is defined by combining two or more events.

**DEFINITION**

Suppose that the marketing director of *Sports Illustrated* believed that anyone who possessed an income greater than \$50,000 or subscribed to more than one other sports magazine could potentially be a good prospect for a direct mail marketing campaign.

Let the event

$$A = \{\text{annual income is greater than } \$50,000\}$$

and

$$B = \{\text{subscribes to more than one other sports magazine}\}.$$

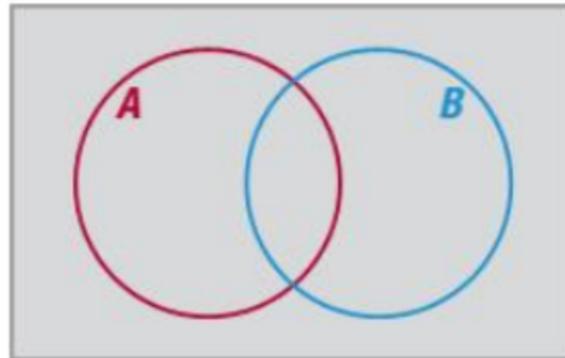


Figure 6.2.1

## Union

The **union** of the events  $A$  and  $B$  is the set of outcomes that are included in event  $A$  or event  $B$  or both. Symbolically, the union of  $A$  and  $B$  is denoted  $A \cup B$  and is read “ $A$  union  $B$ .”

DEFINITION

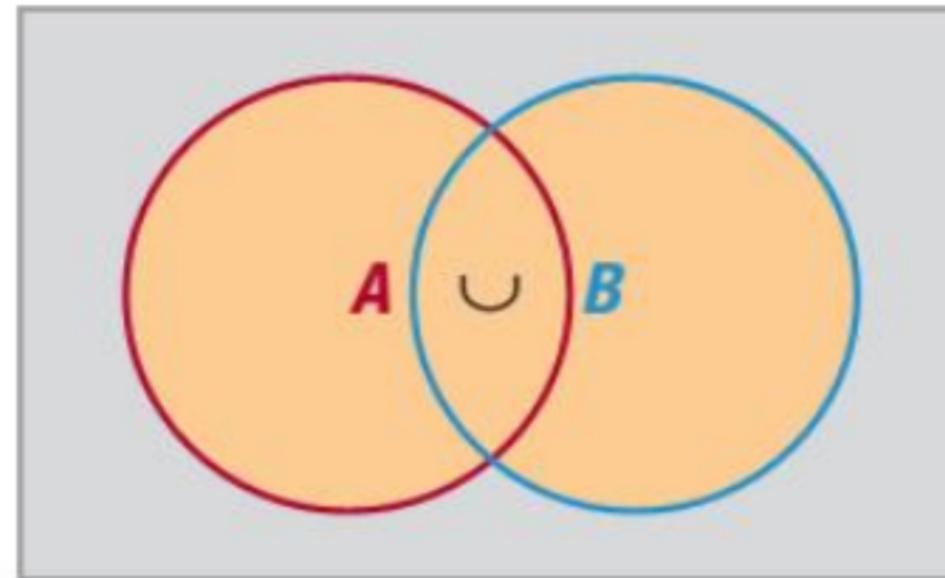


Figure 6.2.2

$A$  or  $B$   
 $A \cup B$

## Intersection

The **intersection** of the events  $A$  and  $B$  is the set of all outcomes that are included in both  $A$  and  $B$ . Symbolically, the intersection of  $A$  and  $B$  is denoted  $A \cap B$  and is read “ $A$  intersect  $B$ .”

DEFINITION

Suppose the marketing director was interested in persons who possessed an annual income greater than \$50,000 and subscribed to more than one other sports magazine. That set would be called the intersection of  $A$  and  $B$ .

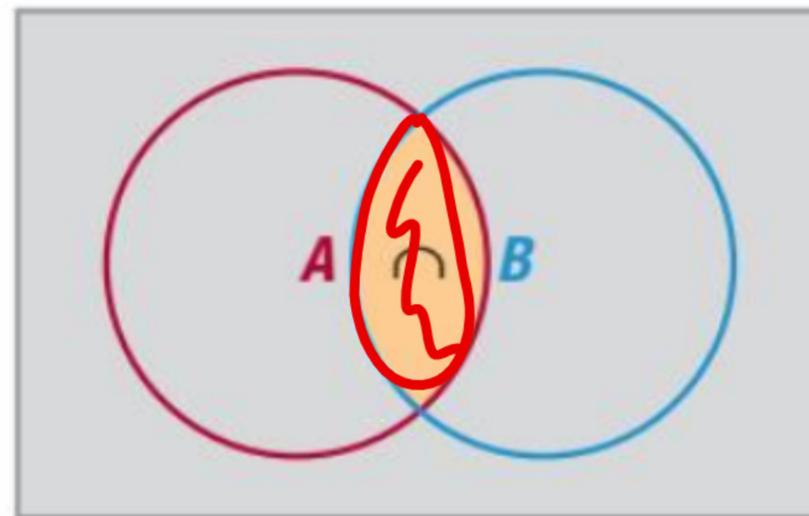


Figure 6.2.3

$A$  and  $B$   
 $A \cap B$

## Complement

The **complement** of an event  $A$  is the set of all outcomes in the sample space that are not in  $A$ . Symbolically, the complement of the set  $A$  will be written as  $A^c$ .

**DEFINITION**

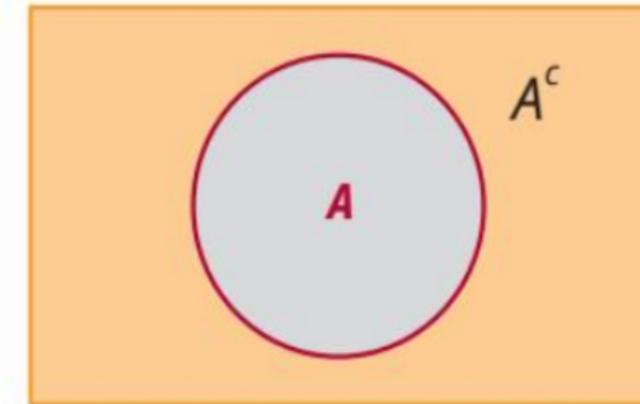


Figure 6.2.4

## Probability Law 5: Complement of an Event

The probability of  $A^c$  is given by  $P(A^c) = 1 - P(A)$ .

**DEFINITION**

In the game of American roulette, the roulette wheel contains the numbers 1 through 36, alternating between red and black. There are two green spaces numbered 0 and 00.

- a. Calculate the probability of the roulette ball landing on a red pocket.  $18/38$
- b. Calculate the probability of the roulette ball not landing on a red pocket.  $20/38$
- c. Calculate the odds in favor of the roulette ball landing on red.

00		3	6	9	12	15	18	21	24	27	30	33	36	2 to 1
		2	5	8	11	14	17	20	23	26	29	32	35	2 to 1
0		1	4	7	10	13	16	19	22	25	28	31	34	2 to 1
1st 12				2nd 12				3rd 12						
1 to 18		EVEN		RED		BLACK		ODD		19 to 36				

The odds against an event  $A$  occurring is given by  $\frac{P(\bar{A})}{P(A)} = \frac{1 - P(A)}{P(A)}$ .

DEFINITION

$\frac{18}{20}$  odds in favor

13	spades
39	

$\frac{1}{3}$

## Mutually Exclusive

Two events are **mutually exclusive** if they have no outcomes in common.

DEFINITION

Mutual exclusivity is also called **disjointness**. Figure 6.2.5 represents two **disjoint** events.

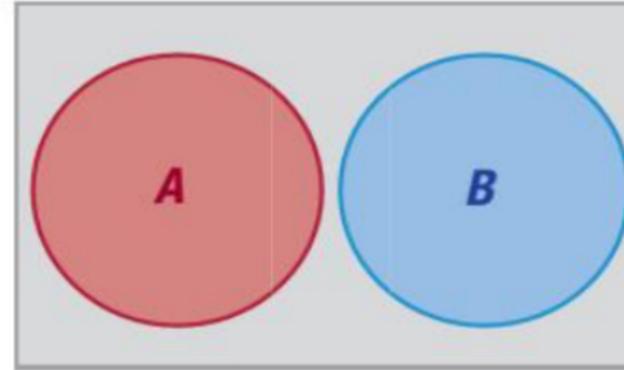


Figure 6.2.5

## Probability Law 6: Union of Mutually Exclusive Events

If the events  $A$  and  $B$  are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

DEFINITION

## Probability Law 7: Intersection of Mutually Exclusive Events

If the events  $A$  and  $B$  are mutually exclusive, then

$$P(A \cap B) = 0.$$

DEFINITION

### Example 6.2.4

---

Suppose that  $P(A) = 0.27$  and the  $P(B) = 0.19$ . If  $A$  and  $B$  are mutually exclusive, what is the probability of  $A \cup B$ ?

# Probability Law 8: The General Addition Rule

For any two events  $A$  and  $B$ ,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

DEFINITION

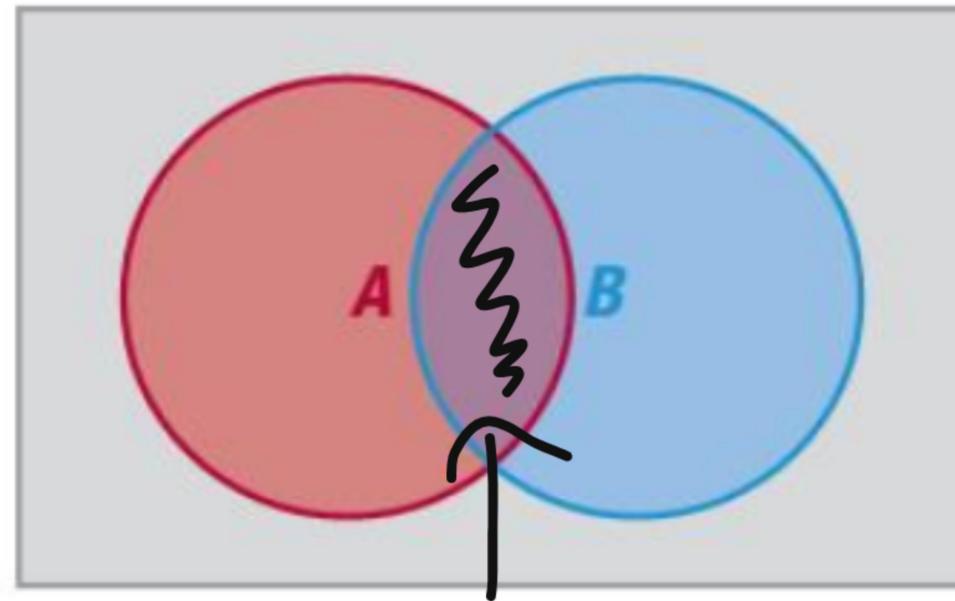


Figure 6.2.6

.30

$$P(A) = .80$$

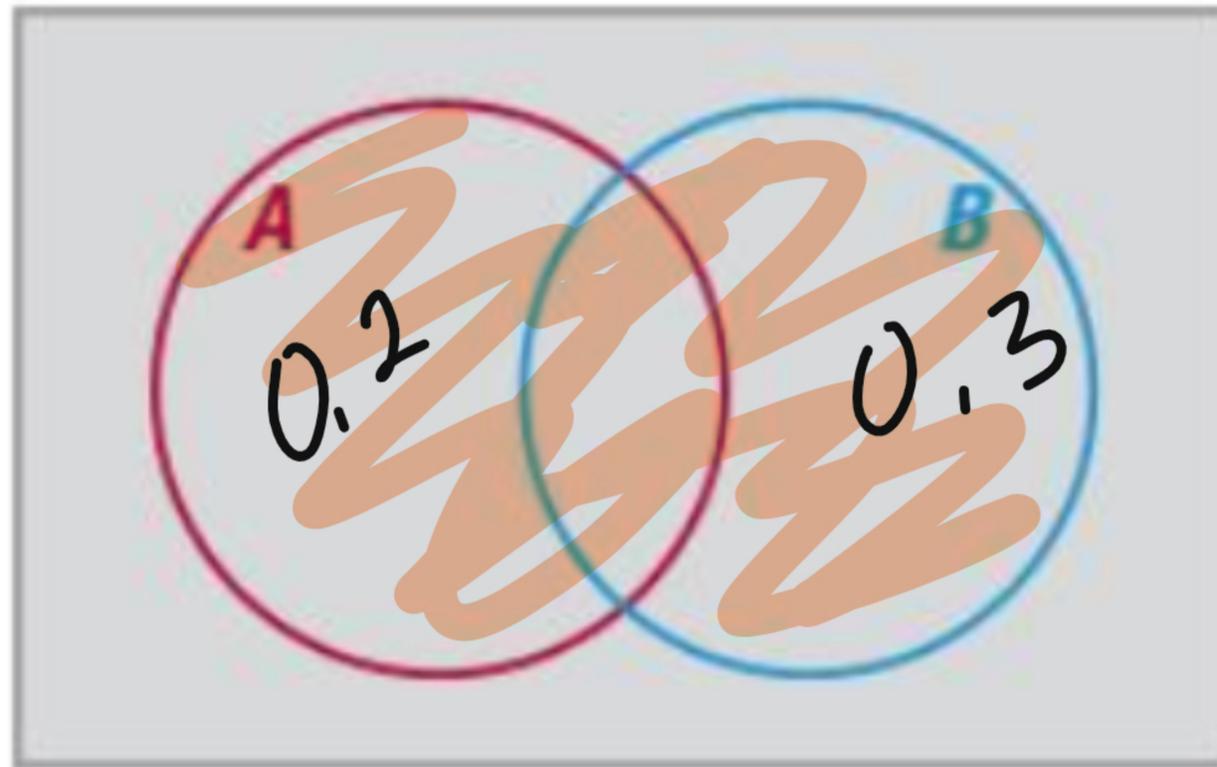
$$P(B) = .40$$

---


$$1.20$$

$$P(A \cap B) = \frac{.30}{.90} P(A \cup B)$$

Suppose that the marketing manager mentioned earlier believed that the probability that someone earns more than \$50,000 is 0.2 and the probability that someone will subscribe to more than one sports magazine is 0.3. If the probability of finding someone in both categories is 0.08, what is the probability of finding someone who is earning over \$50,000 or subscribes to more than one sports magazine?



$$\begin{aligned} P(A) &= 0.2 \\ P(B) &= 0.3 \\ P(A \cap B) &= 0.08 \\ P(A \cup B) &= 0.42 \\ 1 - 0.42 &= 0.58 \end{aligned}$$

		Amount of Life Insurance on Husband (\$)			
		0-50,000	50,000-100,000	100,000-150,000	More than 150,000
Amount of Life Insurance on Wife (\$)	0-50,000	400	200	50	50
	50,000-100,000	50	50	30	30
	100,000-150,000	20	10	25	25
	More than 150,000	20	10	15	15

$$A = \frac{270}{1000} = 27\%$$

$$B = \frac{80}{1000} = 8\%$$

$$P(H > 150) = \frac{120}{1000}$$

$$P(W > 150) = \frac{60}{1000}$$

$$C = \frac{165}{1000} = 16.5\%$$

## Life Insurance Coverage

		Amount of Life Insurance on Husband (\$)			
		0-50,000	50,000-100,000	100,000-150,000	More than 150,000
Amount of Life Insurance on Wife (\$)	0-50,000	400	200	50	50
	50,000-100,000	50	50	30	30
	100,000-150,000	20	10	25	25
	More than 150,000	20	10	15	15

$$1000 - 490 = 510 = \frac{510}{1000} = 51\%$$

- a. For a randomly selected policy, what is the probability that the husband will have between \$50,000 and \$100,000 of insurance?
- b. For a randomly selected policy, what is the probability that the wife will have between \$100,000 and \$150,000 of insurance?
- c. For a randomly selected policy, what is the probability that the wife will have more than \$150,000 of insurance or the husband will have more than \$150,000 of insurance?

- d.** For a randomly selected policy, what is the probability that the wife will have between \$0 and \$50,000 of insurance and the husband will have between \$0 and \$50,000 of insurance?
- e.** For a randomly selected policy, what is the probability that the wife will not have between \$0 and \$50,000 of insurance?
- f.** For a randomly selected policy, what is the probability that the husband will have more than \$50,000 of insurance?
- g.** What approach to probability did you use to calculate your answers?
- h.** Are the events {the wife has more than \$150,000 in insurance} and {the husband has between \$50,000 and \$100,000 of insurance} mutually exclusive? Explain.