Properties of the Definite Integral

When we defined the definite integral $\int_a^b f(x)\,dx$, we implicitly assumed that $a < b$. But the definition as a limit of Riemann sums makes sense even if $a > b$. Notice that if we interchange a and b, then Δx changes from $(b - a)/n$ to $(a - b)/n$. Therefore

$$\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx}$$

If $a = b$, then $\Delta x = 0$ and so

$$\boxed{\int_a^a f(x)\,dx = 0}$$

$\int_a^b$

b ← Interval end

∫ ← Integral sign

a ↑ Intaral start

Properties of the Integral

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant

2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

$4 + 3x\ )\,dx$

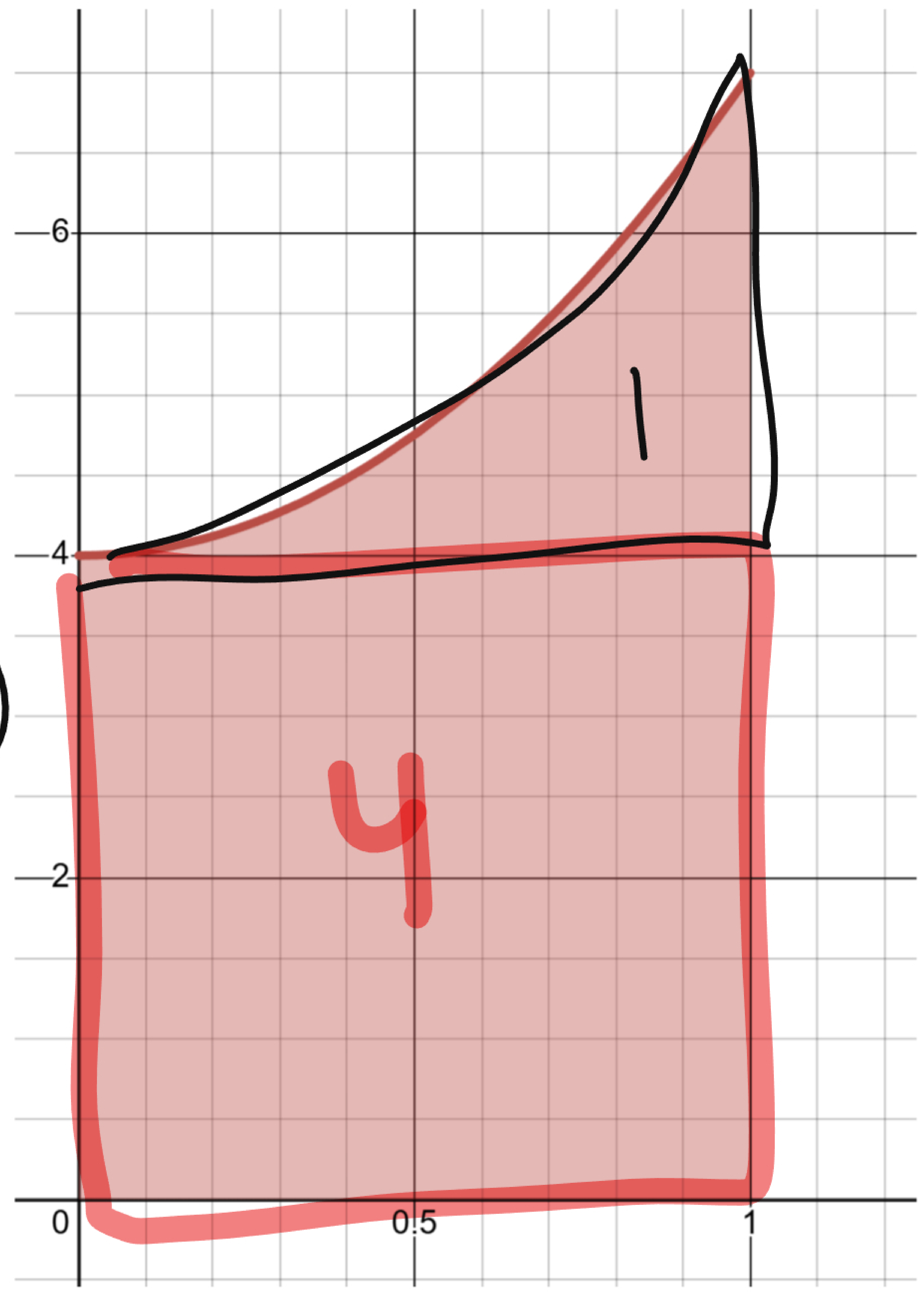
↑ derivate of x

$$4\,dx \quad + \quad \int_0^1 3x^2\,dx$$

$$4x\Big|_0^1 \quad + \quad x^3\Big|_0^1 \quad = 5$$

$$\big(4(1) - 4(0)\big) \quad + \quad \big((1)^3 - (0)^3\big)$$

$$4 \quad + \quad 1$$



$$\int_{-1}^{2} (4x^2 + x + 2)\, dx$$

$$\frac{4}{3}x^3 + \frac{x^2}{2} + 2x \Big|_{-1}^{2}$$

$$\left(\frac{4}{3}(2)^3 + \frac{1}{2}(2)^2 + 2(2)\right) - \left(\frac{4}{3}(-1)^3 + \frac{1}{2}(-1)^2 + 2(-1)\right)$$

$$\left(\frac{32}{3} + 2 + 4\right) - \left(-\frac{4}{3} + \frac{1}{2} - 2\right) = 12 - \frac{1}{2} + 8$$

$$+\frac{4}{3} - \frac{1}{2} + 2 \qquad = \boxed{\frac{39}{2}}$$

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Find the derivative of the function $g(x) = \int_0^x \sqrt{1 + t^2}\, dt$.

$$\frac{d}{dx} \int_0^x \sqrt{1+t^2}\ dt$$

$$g'(x) = \sqrt{1+x^2}$$

Find $\dfrac{d}{dx}\displaystyle\int_1^{x^4} \sec t \, dt.$

$$\boxed{\sec(x^4) \cdot 4x^3}$$

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function F such that $F' = f$.

Evaluate the integral $\int_1^3 e^x \, dx$.

$$e^x \Big|_1^3 = \boxed{e^3 - e}$$

$$\int_0^3 e^x \, dx = e^x \Big|_0^3 = e^3 - 1$$

Evaluate $\int_3^6 \frac{dx}{x}$. $= \int_3^6 \frac{1}{x}\,dx$

$\ln(x)\Big|_3^6 = \boxed{\ln(6) - \ln(3)}$

$\ln\left(\frac{6}{3}\right) = \ln(2)$

$\ln(xy) = \ln(x) + \ln(y)$
$\ln(x/y) = \ln(x) - \ln(y)$
$\ln(x^y) = y\ln(x)$
$\ln(e) = 1$
$\ln(1) = 0$
$\ln(1/x) = -\ln(x)$

Find the area under the cosine curve from 0 to b, where $\boxed{0 \leqslant b \leqslant \pi/2}$.

$$\int_0^b \cos x \, dx$$

$$\frac{d}{dx} \downarrow \quad \begin{array}{c} \sin x \\ \cos x \\ -\sin x \\ -\cos x \end{array} \quad \uparrow \int$$

$$\sin x \Big|_0^b$$

$$\sin(b) - \sin(0) = \sin(b)$$

What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \left.\frac{x^{-1}}{-1}\right]_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$$

$$\downarrow$$

$$-\frac{1}{x} \qquad \left(\frac{-1}{3}\right) - \left(\frac{-1}{-1}\right)$$

1 Table of Indefinite Integrals

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

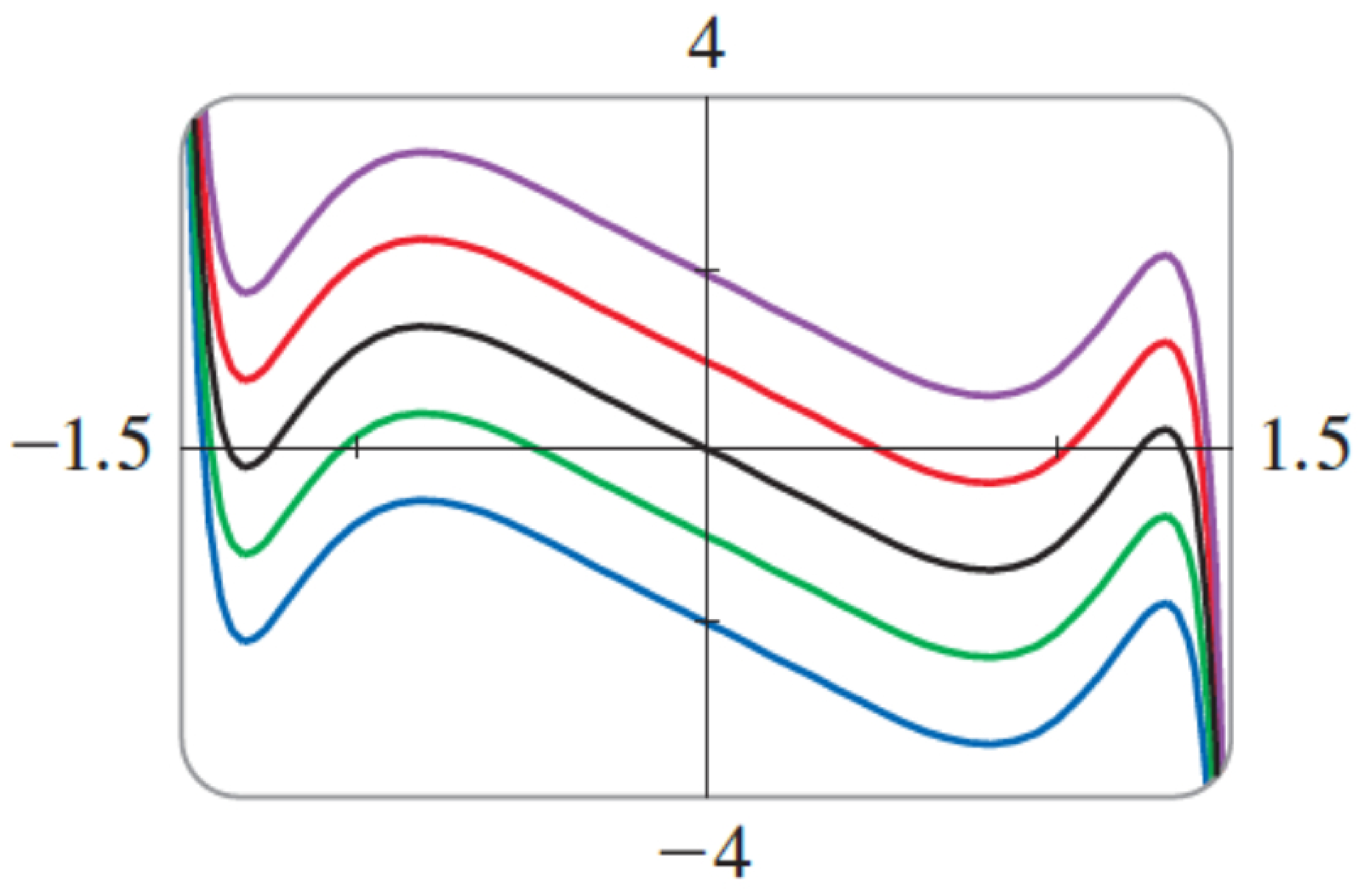
$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\,dx$$



$$\frac{10x^5}{5} - 2\tan x$$

$$2x^5 - 2\tan x + C$$

Evaluate $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$

$\cancel{\int \sin x\,dx = -\cos x + C}$ $\qquad$ $\cancel{\int \cos x\,dx = \sin x + C}$

$\int \sec^2 x\,dx = \tan x + C$ $\qquad$ $\int \csc^2 x\,dx = -\cot x + C$

$\int \underline{\sec x \tan x}\,dx = \sec x + C$ $\qquad$ $\int \underline{\csc x \cot x}\,dx = -\csc x + C$

$$\int \frac{\cos\theta}{\sin\theta}\,\frac{1}{\sin\theta}\,d\theta$$

$$\int \csc\theta\,\cot\theta\,d\theta = \boxed{-\csc\theta + C}$$

$$\frac{\sqrt{t}-1}{t^2}\,dt.$$

$$t^{1/2} - t^{-2}$$

$$\frac{t^{3/2}}{3/2} - \frac{t^{-1}}{-1}$$

$$2t + \frac{2}{3}t^{3/2} + \frac{1}{t}\Big|_1^9$$

$$\left(2(9) + \frac{2}{3}(9)^{3/2} + \frac{1}{9}\right) - \left(2(1) + \frac{2}{3}(1)^{3/2} + \frac{1}{1}\right)$$

$$18 + 18 + \frac{1}{9} - 2 - \frac{2}{3} - 1$$

$$33 - \frac{5}{9} = 32\frac{4}{9}$$

$\sin \pi - \sin 0$

$0 - 0 = 0$

$dx = \int_0^{\pi/2} \cos x \, dx = \sin x \Big|_0^{\pi/2} =$ $\sin \pi/2 - \sin 0$

$1 - 0 = 1$

$\int_{\pi/2}^{\pi} -\cos x \, dx = -\sin x \Big|_{\pi/2}^{\pi}$ $-\sin \pi - (-\sin \pi/2)$

$0 + 1 = 1$

$1 + 1 = 2$

EXAMPLE 6 A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \leqslant t \leqslant 4$. Left 4.5 m

(b) Find the distance traveled during this time period.

$$\int_1^4 t^2 - t - 6\,dt = \frac{t^3}{3} - \frac{t^2}{2} - 6t \Big|_1^4$$

$$= \left(\frac{64}{3} - 8 - 24\right) - \left(\frac{1}{3} - \frac{1}{2} - 6\right)$$

$$= \frac{63}{3} + \frac{1}{2} - 26 = \boxed{\frac{-9}{2}}$$

$t = 3 \qquad t = -2$

dt

$$\begin{array}{ccccc} | & 2 & | & 7/2 & | \\ 1 & - & 3 & + & 4 \end{array}$$

dt

$$\frac{t^3}{3} - \frac{t^2}{2} - 6t \Big|_3^1 \rightarrow \left(\frac{1}{3} - \frac{1}{2} - 6\right) - \left(9 - \frac{9}{2} - 18\right)$$

t

$$\frac{t^3}{3} - \frac{t^2}{2} - 6t \Big|_3^4 \rightarrow \left(\frac{64}{3} - 8 - 24\right) - \left(9 - \frac{9}{2} - 18\right)$$

$$= \boxed{\frac{61}{6}} \approx \boxed{10.17\,m}$$