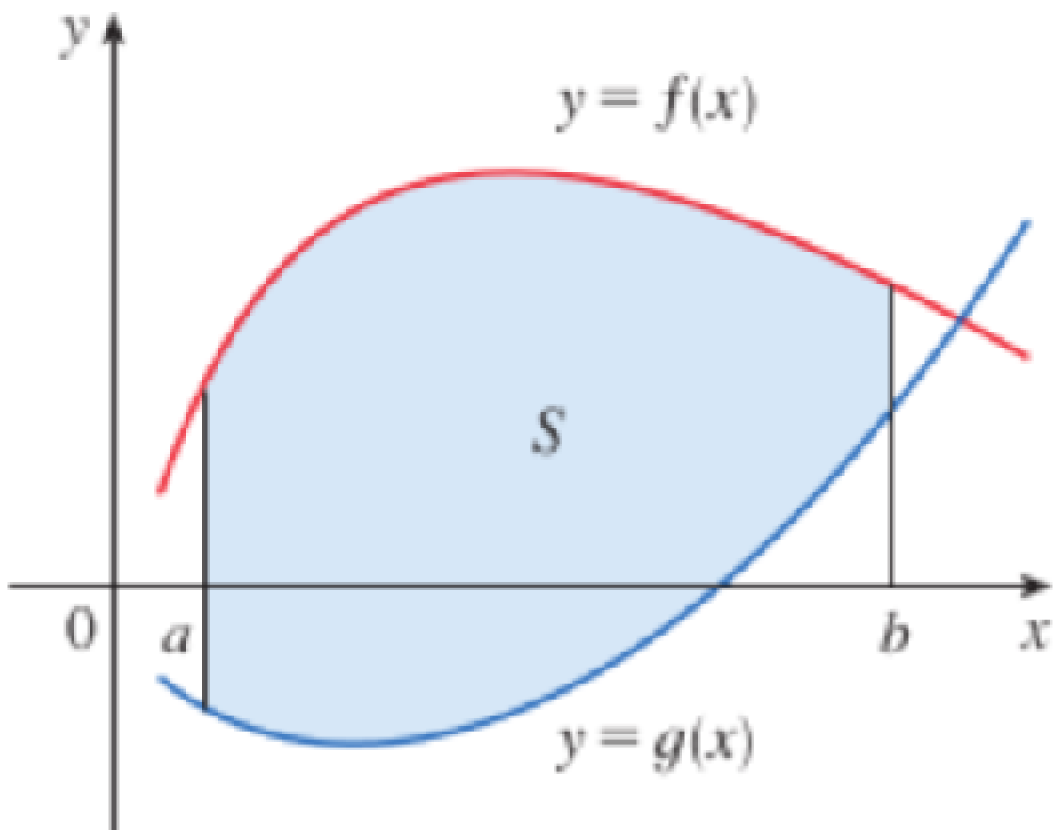
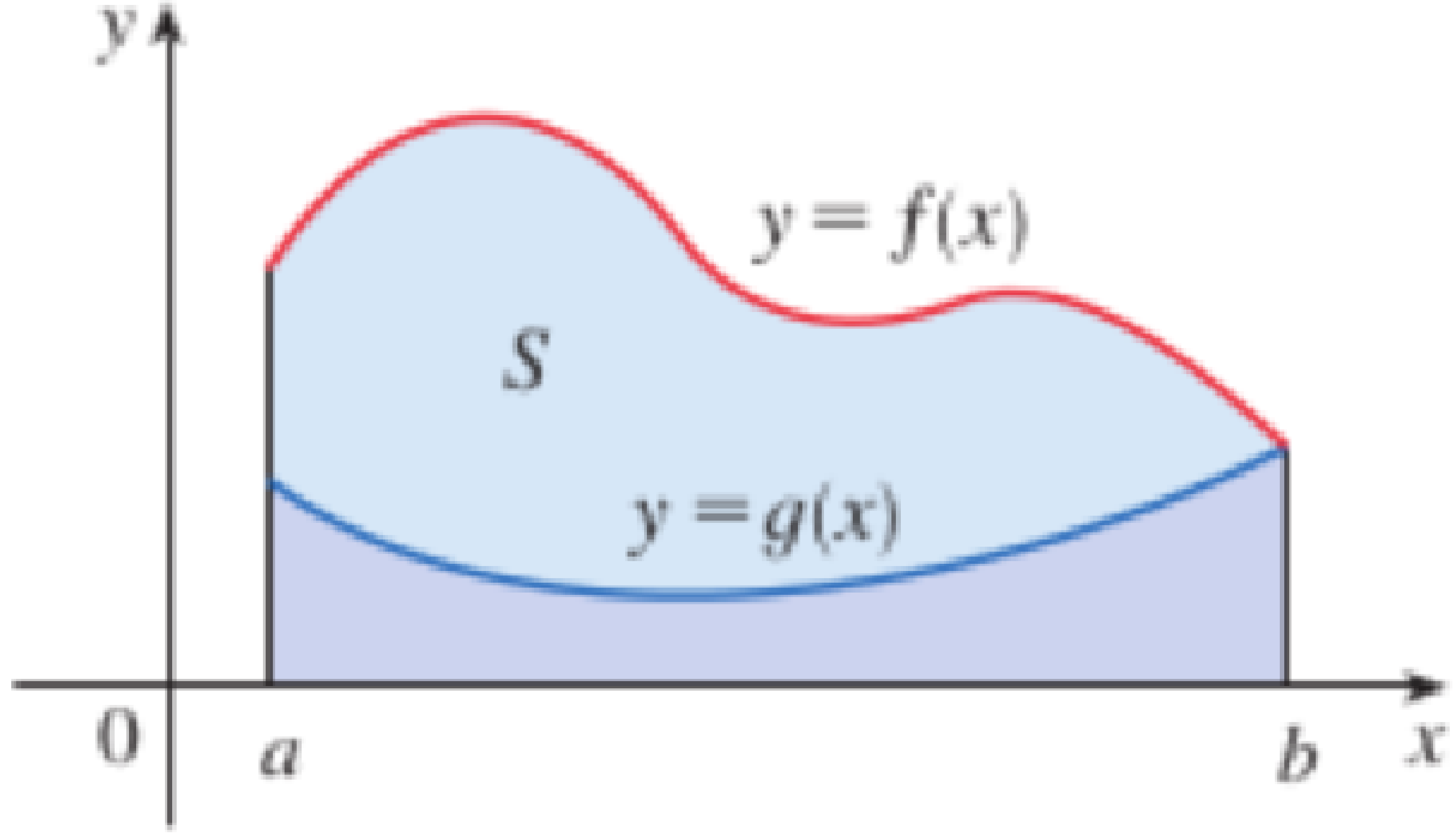
6.1. Areas between Curves

Area between Curves: Integrating with Respect to x

Consider the region S shown in Figure 1 that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geqslant g(x)$ for all x in $[a, b]$.

Figure 1





▸ Details

$$A = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx$$

$$A = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

$$= \int_a^b f(x) - g(x)\, dx$$

Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

$y = e^x \quad (0, 1)$

$y = x$

$$A = e - \frac{3}{2}$$

$$\int_0^1 e^x \, dx - \int_0^1 x \, dx$$

$$e^x \Big]_0^1 - \frac{x^2}{2} \Big]_0^1 = (e - 1) - \left(\frac{1}{2} - 0\right)$$

Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

$y = x^2$ $\qquad$ $y = 2x - x^2$

$x^2 = 2x - x^2$

$2x^2 - 2x = 0$

$2x(x-1) = 0$

$2x = 0$ $\qquad$ $x - 1 = 0$

$x = 0$ $\qquad$ $x = 1$

$\int_0^1 (2x - x^2) - (x^2)\,dx$

$\int_0^1 2x - 2x^2\,dx$

$\frac{2x^2}{2} - \frac{2x^3}{3}$

$x^2 - \frac{2}{3}x^3 \Big]_0^1 = 1 - \frac{2}{3} = \boxed{\frac{1}{3}}$

Find the approximate area of the region bounded by the curves $y = x/\sqrt{x^2+1}$ and $y = x^4 - x$.

$$y = \frac{x}{\sqrt{x^2+1}}$$

$$y = x^4 - x$$

$$\int_0^{1.181} \frac{x}{\sqrt{x^2+1}}\,dx - \int_0^{1.181} x^4 - x\,dx$$

$$u = x^2+1$$

$$du = 2x\,dx$$

$$\frac{1}{2}du = x\,dx$$

$$\int_0^{1.181} \frac{1}{\sqrt{u}} \cdot \frac{1}{2}\,du$$

$$\int_0$$

Find the approximate area of the region bounded by the curves $y = x/\sqrt{x^2+1}$ and $y = x^4 - x$.

$$y = \frac{x}{\sqrt{x^2+1}}$$

$$\int_0^{1.181} \frac{x}{\sqrt{x^2+1}}\,dx - \int_0^{1.181} x^4 - x\,dx$$

$$y = x^4 - x$$

$$\frac{1}{2}\int_0^{1.181} u^{-1/2}\,du$$

$$\frac{1}{2}\left(\frac{u^{1/2}}{1/2}\right) = \sqrt{u} = \sqrt{x^2+1}\,\Big]_0^{1.181}$$

Find the approximate area of the region bounded by the curves $y = x/\sqrt{x^2+1}$ and $y = x^4 - x$.

$y = \frac{x}{\sqrt{x^2+1}}$

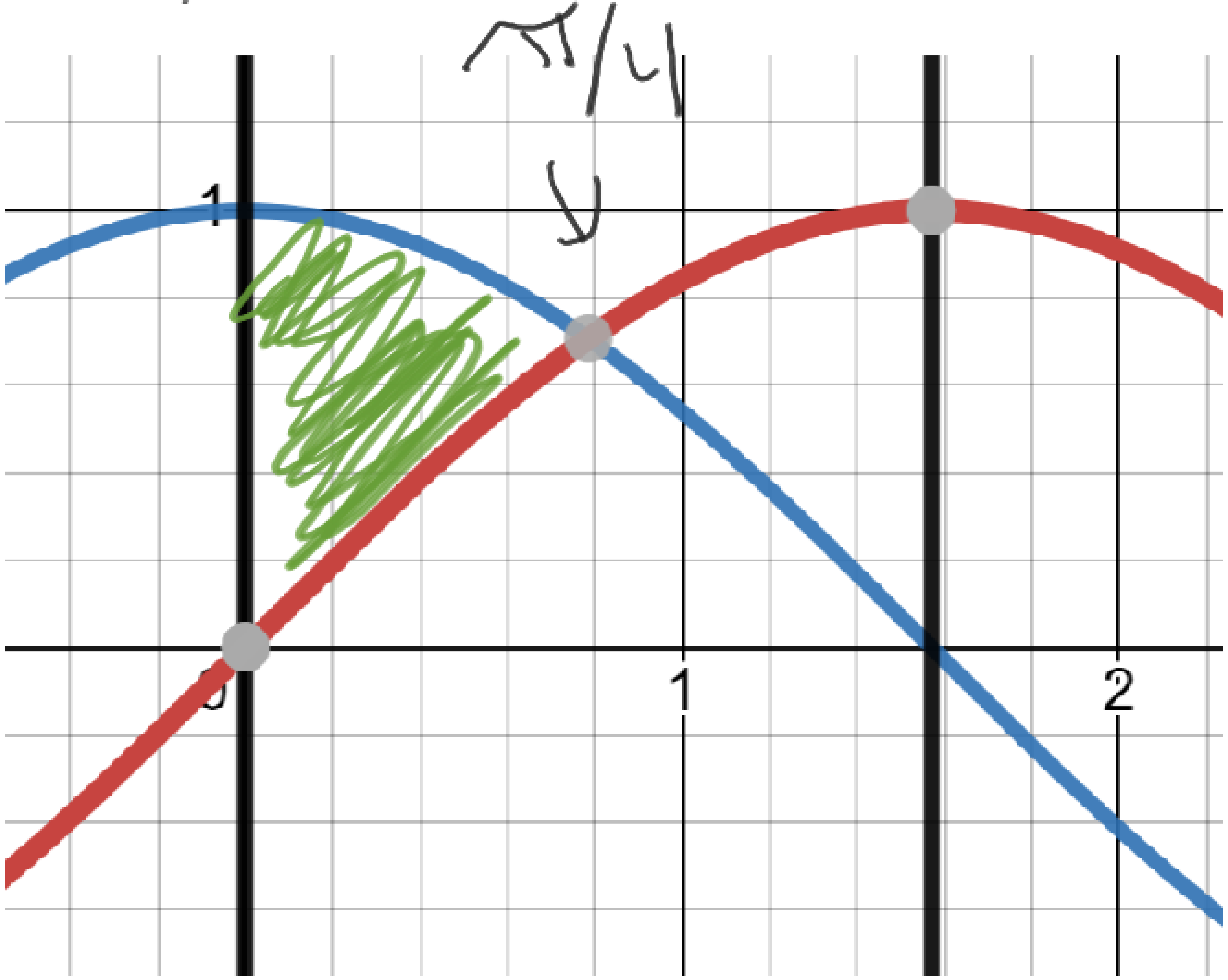
$y = x^4 - x$

$\int_0^{1.181} \frac{x}{\sqrt{x^2+1}}\,dx - \int_0^{1.181} x^4 - x\,dx$

$\sqrt{x^2+1}\Big]_0^{1.181} - \left(\frac{x^5}{5} - \frac{x^2}{2}\right)\Big]_0^{1.18}$

$1.785 - 1 = .785$

Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.



$$\int_0^{\pi/4} \cos x - \sin x \, dx$$

$$\left. \sin x - (-\cos x) \right]_0^{\pi/4}$$

$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\quad \sin(0) = 0$

$\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\quad \cos(0) = 1$

$$\left(\frac{\sqrt{2}}{2} - 0\right) + \left(\frac{\sqrt{2}}{2} - 1\right) = \boxed{\sqrt{2} - 1}$$

Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

$\pi/4$

$\boxed{2\sqrt{2} - 2}$

$$\int_{\pi/4}^{\pi/2} \sin x - \cos x$$

$$-\cos x - \sin x \Big]_{\pi/4}^{\pi/2}$$

$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\qquad$ $\sin\left(\frac{\pi}{2}\right) = 1$

$\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\qquad$ $\cos\left(\frac{\pi}{2}\right) = 0$

$$-\left(0 - \frac{\sqrt{2}}{2}\right) - \left(1 - \frac{\sqrt{2}}{2}\right) = \boxed{\sqrt{2} - 1}$$

$\sin x$

$\cos x$

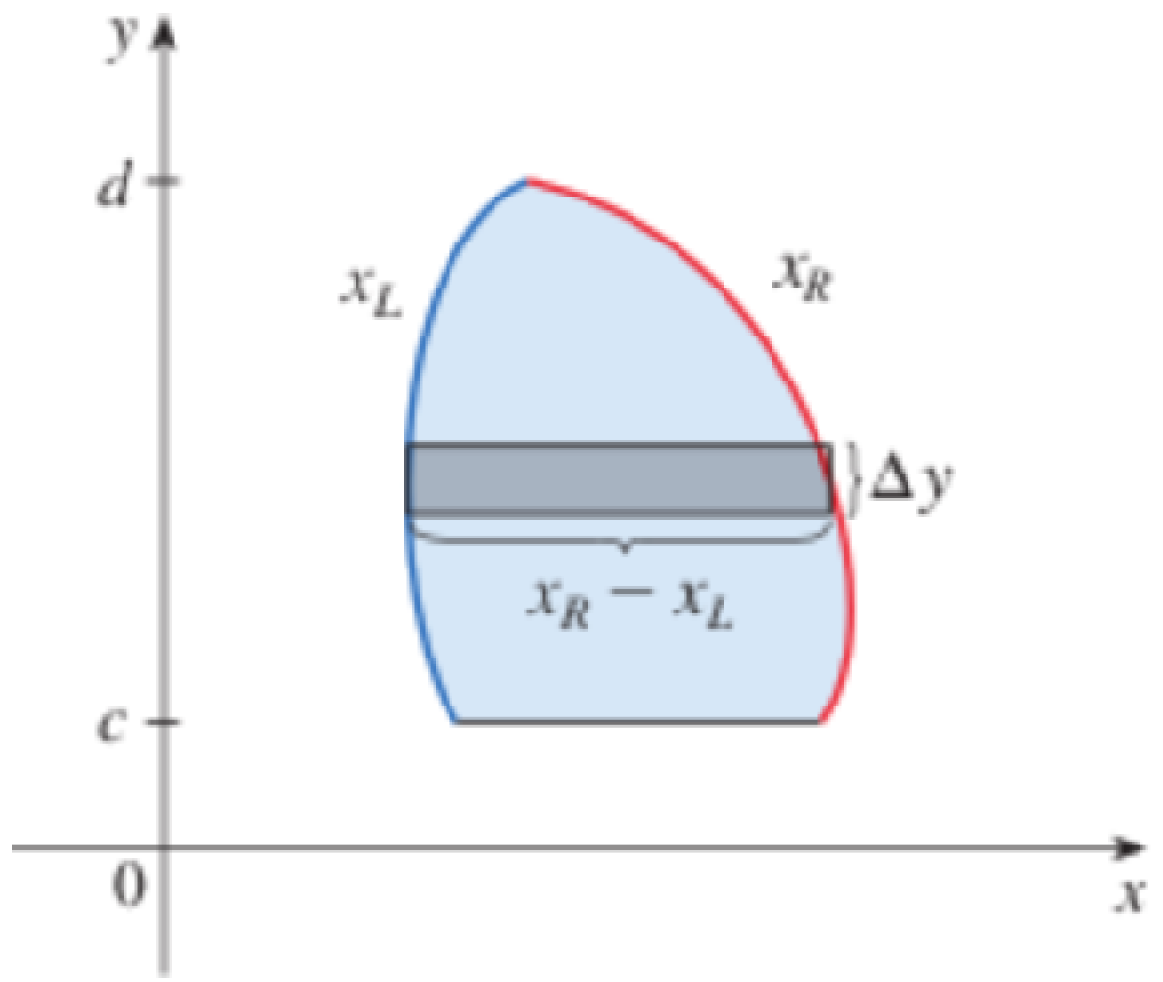
$-\sin x$

$-\cos x$

$$A = \int_c^d (x_R - x_L)\, dy$$

Here a typical approximating rectangle has dimensions $x_R - x_L$ and Δy.

Figure 11



Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

$y = x - 1$

$y + 1 = x$

$y^2 = 2x + 6$

$y^2 - 6 = 2x$

$\frac{y^2 - 6}{2} = x$

$1/2 y^2 - 3 = x$

$$\int_{-2}^{4} (y+1) - (1/2 y^2 - 3)\, dy$$

Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

$$\int_{-2}^{4} -\frac{1}{2}y^2 + y + 4 \, dy$$

$$-\frac{1}{2}\left(\frac{y^3}{3}\right) + \frac{y^2}{2} + 4y \Big]_{-2}^{4}$$

$$\left[-\frac{(4)^3}{6} + \frac{(4)^2}{2} + 4(4)\right] - \left[-\frac{(-2)^3}{6} + \frac{(-2)^2}{2} + 4(-2)\right]$$

$$-\frac{64}{6} + 8 + 16 - \frac{8}{6} - 2 + 8 = \frac{-72}{6} = \boxed{18}$$

$$= -12 + 30$$

Find the area of the region enclosed by the curves $y = 1/x$, $y = x$, and $y = \frac{1}{4}x$, using

a. x as the variable of integration and

b. y as the variable of integration.