

10

Parametric Equations and Polar Coordinates



10.2

Calculus with Parametric Curves



Tangents

Tangents

Suppose f and g are differentiable functions and we want to find the tangent line at a point on the parametric curve $x = f(t)$, $y = g(t)$, where y is also a differentiable function of x .

Then the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Tangents

If $dx/dt \neq 0$, we can solve for dy/dx :

1

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if } \frac{dx}{dt} \neq 0$$

Equation 1 (which you can remember by thinking of canceling the dt 's) enables us to find the slope dy/dx of the tangent to a parametric curve without having to eliminate the parameter t .

Tangents

We see from (1) that the curve has a horizontal tangent when $dy/dt = 0$ (provided that $dx/dt \neq 0$) and it has a vertical tangent when $dx/dt = 0$ (provided that $dy/dt \neq 0$).

This information is useful for sketching parametric curves.

It is also useful to consider d^2y/dx^2 . This can be found by replacing y by dy/dx in Equation 1:

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}}$$

Example 1

A curve C is defined by the parametric equations $x = t^2$,
 $y = t^3 - 3t$.

- (a) Show that C has two tangents at the point $(3, 0)$ and find their equations.
- (b) Find the points on C where the tangent is horizontal or vertical.
- (c) Determine where the curve is concave upward or downward.
- (d) Sketch the curve.

Example 1 – *Solution*

(a) Notice that $y = t^3 - 3t = t(t^2 - 3) = 0$ when $t = 0$ or $t = \pm\sqrt{3}$.
Therefore the point $(3, 0)$ on C arises from two values of the parameter, $t = \sqrt{3}$ and $t = -\sqrt{3}$.

This indicates that C crosses itself at $(3, 0)$.

Example 1 – Solution

cont'd

Since

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t} \\ &= \frac{3}{2} \left(t - \frac{1}{t} \right)\end{aligned}$$

the slope of the tangent when $t = \pm\sqrt{3}$ is $dy/dx = \pm 6/(2\sqrt{3}) = \pm\sqrt{3}$, so the equations of the tangents at $(3, 0)$ are

$$y = \sqrt{3}(x - 3) \quad \text{and} \quad y = -\sqrt{3}(x - 3)$$

Example 1 – Solution

cont'd

(b) C has a horizontal tangent when $dy/dx = 0$, that is, when $dy/dt = 0$ and $dx/dt \neq 0$. Since $dy/dt = 3t^2 - 3$, this happens when $t^2 = 1$, that is, $t = \pm 1$.

The corresponding points on C are $(1, -2)$ and $(1, 2)$.

C has a vertical tangent when $dx/dt = 2t = 0$, that is, $t = 0$. (Note that $dy/dt \neq 0$ there.)

The corresponding point on C is $(0, 0)$.

Example 1 – Solution

cont'd

(c) To determine concavity we calculate the second derivative:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{\frac{3}{2} \left(1 + \frac{1}{t^2} \right)}{2t} \\ &= \frac{3(t^2 + 1)}{4t^3}\end{aligned}$$

Thus the curve is concave upward when $t > 0$ and concave downward when $t < 0$.

Example 1 – Solution

cont'd

(d) Using the information from parts (b) and (c), we sketch C in Figure 1.

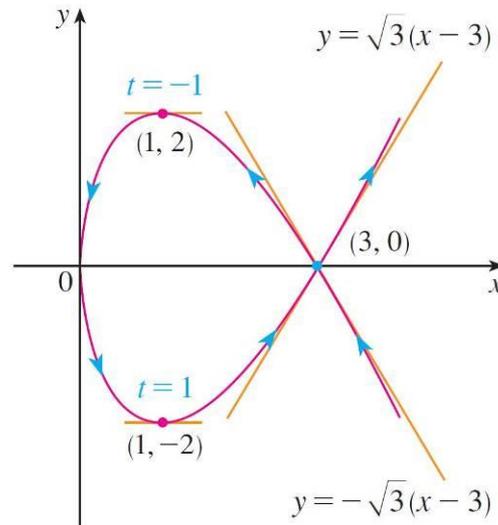


Figure 1



Areas

Areas

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x) dx$, where $F(x) \geq 0$.

If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_a^b y dx = \int_{\alpha}^{\beta} g(t)f'(t) dt \quad \left[\text{or} \quad \int_{\beta}^{\alpha} g(t)f'(t) dt \right]$$

Example 3

Find the area under one arch of the cycloid

$$x = r(\theta - \sin \theta) \quad y = r(1 - \cos \theta)$$

(See Figure 3.)

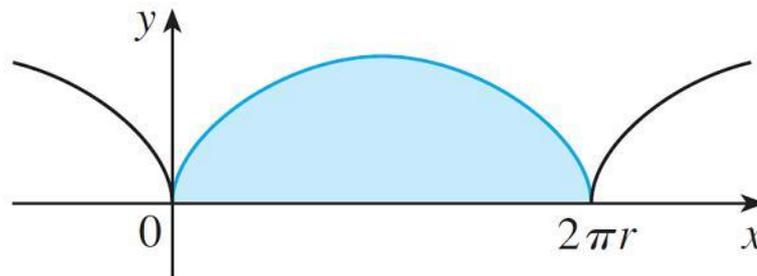


Figure 3

Example 3 – *Solution*

One arch of the cycloid is given by $0 \leq \theta \leq 2\pi$.

Using the Substitution Rule with $y = r(1 - \cos \theta)$ and $dx = r(1 - \cos \theta)d\theta$, we have

$$\begin{aligned} A &= \int_0^{2\pi r} y \, dx = \int_0^{2\pi} r(1 - \cos \theta) r(1 - \cos \theta) \, d\theta \\ &= r^2 \int_0^{2\pi} (1 - \cos \theta)^2 \, d\theta \\ &= r^2 \int_0^{2\pi} (1 - 2 \cos \theta + \cos^2 \theta) \, d\theta \end{aligned}$$

Example 3 – Solution

cont'd

$$= r^2 \int_0^{2\pi} \left[1 - 2 \cos \theta + \frac{1}{2}(1 + \cos 2\theta) \right] d\theta$$

$$= r^2 \left[\frac{3}{2} \theta - 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi}$$

$$= r^2 \left(\frac{3}{2} \cdot 2\pi \right) = 3\pi r^2$$



Arc Length

Arc Length

We already know how to find the length L of a curve C given in the form $y = F(x)$, $a \leq x \leq b$.

If F' is continuous, then

$$\boxed{2} \quad L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Suppose that C can also be described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, where $dx/dt = f'(t) > 0$.

Arc Length

This means that C is traversed once, from left to right, as t increases from α to β and $f(\alpha) = a$, $f(\beta) = b$.

Putting Formula 1 into Formula 2 and using the Substitution Rule, we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2} \frac{dx}{dt} dt$$

Since $dx/dt > 0$, we have

$$\boxed{3} \quad L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Arc Length

Even if C can't be expressed in the form $y = F(x)$, Formula 3 is still valid but we obtain it by polygonal approximations.

We divide the parameter interval $[\alpha, \beta]$ into n subintervals of equal width Δt .

If $t_0, t_1, t_2, \dots, t_n$ are the endpoints of these subintervals, then $x_i = f(t_i)$ and $y_i = g(t_i)$ are the coordinates of points $P_i(x_i, y_i)$ that lie on C and the polygon with vertices P_0, P_1, \dots, P_n approximates C . (See Figure 4.)

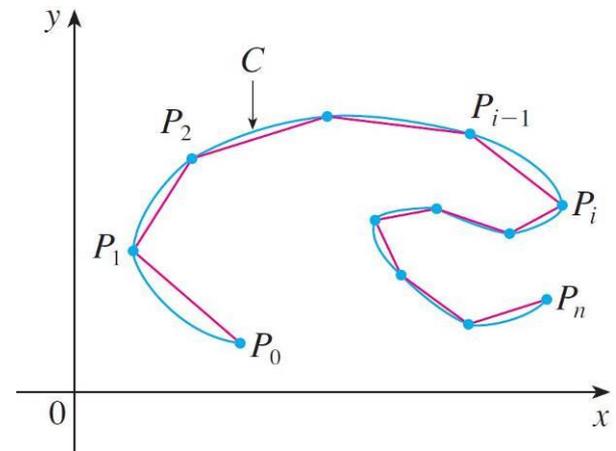


Figure 4

Arc Length

We define the length L of C to be the limit of the lengths of these approximating polygons as $n \rightarrow \infty$:

$$L = \lim_{n \rightarrow \infty} \sum_{i=1}^n |P_{i-1}P_i|$$

The Mean Value Theorem, when applied to f on the interval $[t_{i-1}, t_i]$, gives a number t_i^* in (t_{i-1}, t_i) such that

$$f(t_i) - f(t_{i-1}) = f'(t_i^*) (t_i - t_{i-1})$$

Arc Length

If we let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, this equation becomes

$$\Delta x_i = f'(t_i^*) \Delta t$$

Similarly, when applied to g , the Mean Value Theorem gives a number t_i^{**} in (t_{i-1}, t_i) such that

$$\Delta y_i = g'(t_i^{**}) \Delta t$$

Arc Length

Therefore

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{[f'(t_i^*) \Delta t]^2 + [g'(t_i^{**}) \Delta t]^2} \\ &= \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2} \Delta t \end{aligned}$$

and so

$$\boxed{4} \quad L = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2} \Delta t$$

Arc Length

The sum in (4) resembles a Riemann sum for the function $\sqrt{[f'(t)]^2 + [g'(t)]^2}$ but it is not exactly a Riemann sum because $t_i^* \neq t_i^{**}$ in general.

Nevertheless, if f' and g' are continuous, it can be shown that the limit in (4) is the same as if t_i^* and t_i^{**} were equal, namely,

$$L = \int_{\alpha}^{\beta} \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$$

Arc Length

Thus, using Leibniz notation, we have the following result, which has the same form as Formula 3.

5 Theorem If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β , then the length of C is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Notice that the formula in Theorem 5 is consistent with the general formulas $L = \int ds$ and $(ds)^2 = (dx)^2 + (dy)^2$.

Arc Length

Notice that the integral gives twice the arc length of the circle because as t increases from 0 to 2π , the point $(\sin 2t, \cos 2t)$ traverses the circle twice.

In general, when finding the length of a curve C from a parametric representation, we have to be careful to ensure that C is traversed only once as t increases from α to β .

Example 5

Find the length of one arch of the cycloid $x = r(\theta - \sin \theta)$,
 $y = r(1 - \cos \theta)$.

Solution:

From Example 3 we see that one arch is described by the parameter interval $0 \leq \theta \leq 2\pi$.

Since

$$\frac{dx}{d\theta} = r(1 - \cos \theta) \quad \text{and} \quad \frac{dy}{d\theta} = r \sin \theta$$

Example 5 – Solution

cont'd

We have

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \\ &= \int_0^{2\pi} \sqrt{r^2(1 - \cos \theta)^2 + r^2 \sin^2 \theta} d\theta \\ &= \int_0^{2\pi} \sqrt{r^2(1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta)} d\theta \\ &= r \int_0^{2\pi} \sqrt{2(1 - \cos \theta)} d\theta \end{aligned}$$

Example 5 – Solution

cont'd

To evaluate this integral we use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ with $\theta = 2x$, which gives $1 - \cos \theta = 2 \sin^2(\theta/2)$.

Since $0 \leq \theta \leq 2\pi$, we have $0 \leq \theta/2 \leq \pi$ and so $\sin(\theta/2) \geq 0$.

Therefore

$$\begin{aligned}\sqrt{2(1 - \cos \theta)} &= \sqrt{4 \sin^2(\theta/2)} \\ &= 2 |\sin(\theta/2)| \\ &= 2 \sin(\theta/2)\end{aligned}$$

Example 5 – *Solution*

cont'd

and so

$$\begin{aligned} L &= 2r \int_0^{2\pi} \sin(\theta/2) d\theta \\ &= 2r[-2 \cos(\theta/2)]_0^{2\pi} \\ &= 2r[2 + 2] \\ &= 8r \end{aligned}$$



Surface Area

Surface Area

Suppose the curve c given by the parametric equations $x = f(t), y = g(t), \alpha \leq t \leq \beta$, where f', g' are continuous, $g(t) \geq 0$, is rotated about the x -axis. If C is traversed exactly once as t increases from α to β , then the area of the resulting surface is given by

$$\boxed{6} \quad S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The general symbolic formulas $S = \int 2\pi y ds$ and $S = \int 2\pi x ds$ are still valid, but for parametric curves we use

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Example 6

Show that the surface area of a sphere of radius r is $4\pi r^2$.

Solution:

The sphere is obtained by rotating the semicircle

$$x = r \cos t \quad y = r \sin t \quad 0 \leq t \leq \pi$$

about the x -axis.

Therefore, from Formula 6, we get

$$S = \int_0^{\pi} 2\pi r \sin t \sqrt{(-r \sin t)^2 + (r \cos t)^2} dt$$

Example 6 – Solution

cont'd

$$= 2\pi \int_0^{\pi} r \sin t \sqrt{r^2(\sin^2 t + \cos^2 t)} dt$$

$$= 2\pi \int_0^{\pi} r \sin t \cdot r dt$$

$$= 2\pi r^2 \int_0^{\pi} \sin t dt$$

$$= 2\pi r^2 (-\cos t) \Big|_0^{\pi}$$

$$= 4\pi r^2$$