

# 10

## Parametric Equations and Polar Coordinates



# 10.4

## Areas and Lengths in Polar Coordinates

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# Areas and Lengths in Polar Coordinates

In this section we develop the formula for the area of a region whose boundary is given by a polar equation. We need to use the formula for the area of a sector of a circle:

1

$$A = \frac{1}{2} r^2 \theta$$

where, as in Figure 1,  $r$  is the radius and  $\theta$  is the radian measure of the central angle.

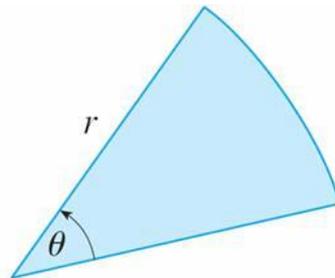


Figure 1

# Areas and Lengths in Polar Coordinates

Formula 1 follows from the fact that the area of a sector is proportional to its central angle:

$$A = (\theta/2\pi)\pi r^2 = \frac{1}{2} r^2 \theta.$$

# Areas and Lengths in Polar Coordinates

Let  $\mathcal{R}$  be the region, illustrated in Figure 2, bounded by the polar curve  $r = f(\theta)$  and by the rays  $\theta = a$  and  $\theta = b$ , where  $f$  is a positive continuous function and where  $0 < b - a \leq 2\pi$ .

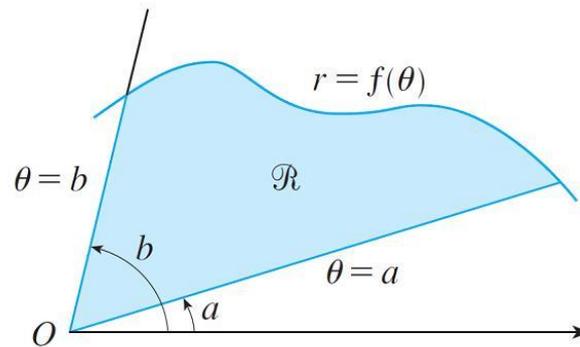


Figure 2

We divide the interval  $[a, b]$  into subintervals with endpoints  $\theta_0, \theta_1, \theta_2, \dots, \theta_n$  and equal width  $\Delta\theta$ .

# Areas and Lengths in Polar Coordinates

The rays  $\theta = \theta_j$  then divide  $\mathcal{R}$  into  $n$  smaller regions with central angle  $\Delta\theta = \theta_j - \theta_{j-1}$ . If we choose  $\theta_j^*$  in the  $j$ th subinterval  $[\theta_{j-1}, \theta_j]$ , then the area  $\Delta A_j$  of the  $j$ th region is approximated by the area of the sector of a circle with central angle  $\Delta\theta$  and radius  $f(\theta_j^*)$ . (See Figure 3.)

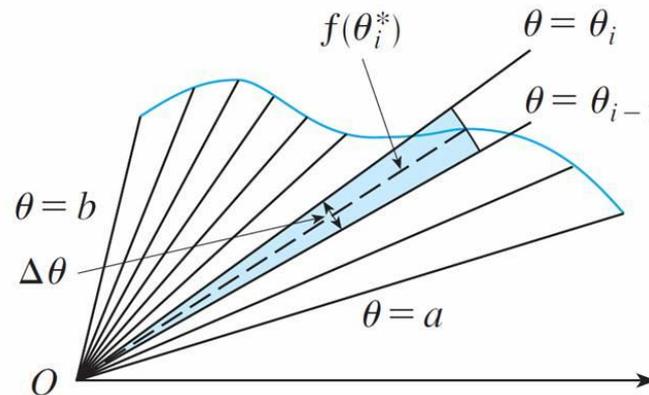


Figure 3

# Areas and Lengths in Polar Coordinates

Thus from Formula 1 we have

$$\Delta A_i \approx \frac{1}{2} [f(\theta_i^*)]^2 \Delta \theta$$

and so an approximation to the total area  $A$  of  $\mathcal{R}$  is

2

$$A \approx \sum_{i=1}^n \frac{1}{2} [f(\theta_i^*)]^2 \Delta \theta$$

It appears from Figure 3 that the approximation in (2) improves as  $n \rightarrow \infty$ .

# Areas and Lengths in Polar Coordinates

But the sums in (2) are Riemann sums for the function  $g(\theta) = \frac{1}{2} [f(\theta)]^2$ , so

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{2} [f(\theta_i^*)]^2 \Delta\theta = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta$$

It therefore appears plausible that the formula for the area  $A$  of the polar region  $\mathcal{R}$  is

3

$$A = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta$$

# Areas and Lengths in Polar Coordinates

Formula 3 is often written as

4

$$A = \int_a^b \frac{1}{2} r^2 d\theta$$

with the understanding that  $r = f(\theta)$ . Note the similarity between Formulas 1 and 4.

When we apply Formula 3 or 4 it is helpful to think of the area as being swept out by a rotating ray through  $O$  that starts with angle  $a$  and ends with angle  $b$ .

# Example 1

Find the area enclosed by one loop of the four-leaved rose  $r = \cos 2\theta$ .

**Solution:**

Notice from Figure 4 that the region enclosed by the right loop is swept out by a ray that rotates from  $\theta = -\pi/4$  to  $\theta = \pi/4$ .

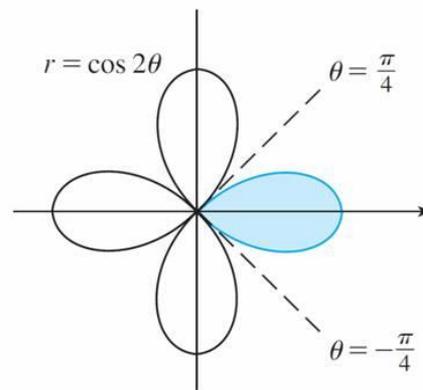


Figure 4

# Example 1 – *Solution*

cont'd

Therefore Formula 4 gives

$$\begin{aligned} A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 d\theta \\ &= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta d\theta \\ &= \int_0^{\pi/4} \cos^2 2\theta d\theta \end{aligned}$$

# Example 1 – *Solution*

cont'd

$$\begin{aligned} A &= \int_0^{\pi/4} \frac{1}{2} (1 + \cos 4\theta) d\theta \\ &= \frac{1}{2} \left[ \theta + \frac{1}{4} \sin 4\theta \right]_0^{\pi/4} \\ &= \frac{\pi}{8} \end{aligned}$$



# Arc Length

# Arc Length

To find the length of a polar curve  $r = f(\theta)$ ,  $a \leq \theta \leq b$ , we regard  $\theta$  as a parameter and write the parametric equations of the curve as

$$x = r \cos \theta = f(\theta) \cos \theta \qquad y = r \sin \theta = f(\theta) \sin \theta$$

Using the Product Rule and differentiating with respect to  $\theta$ , we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta$$

# Arc Length

So, using  $\cos^2 \theta + \sin^2 \theta = 1$ , we have

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \cos^2 \theta - 2r \frac{dr}{d\theta} \cos \theta \sin \theta + r^2 \sin^2 \theta \\ &\quad + \left(\frac{dr}{d\theta}\right)^2 \sin^2 \theta + 2r \frac{dr}{d\theta} \sin \theta \cos \theta + r^2 \cos^2 \theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2\end{aligned}$$

# Arc Length

Assuming that  $f'$  is continuous, we can write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

Therefore the length of a curve with polar equation  $r = f(\theta)$ ,  $a \leq \theta \leq b$ , is

5

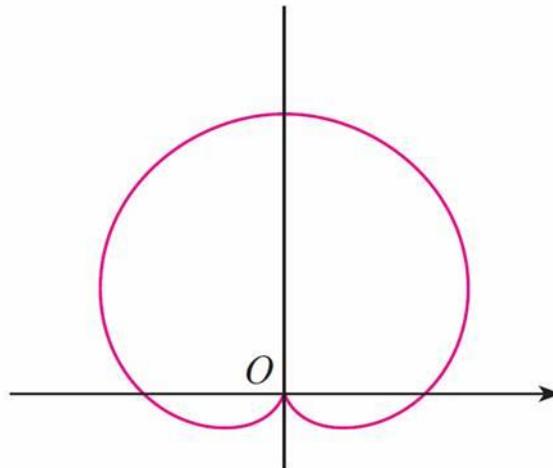
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

# Example 4

Find the length of the cardioid  $r = 1 + \sin \theta$ .

**Solution:**

The cardioid is shown in Figure 8.



$$r = 1 + \sin \theta$$

Figure 8

# Example 4 – Solution

cont'd

5

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Its full length is given by the parameter interval  $0 \leq \theta \leq 2\pi$ , so Formula 5 gives

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \\ &= \int_0^{2\pi} \sqrt{(1 + \sin \theta)^2 + \cos^2 \theta} d\theta \end{aligned}$$

# Example 4 – *Solution*

cont'd

$$= \int_0^{2\pi} \sqrt{2 + 2 \sin \theta} \, d\theta$$

We could evaluate this integral by multiplying and dividing the integrand by  $\sqrt{2 - 2 \sin \theta}$ , or we could use a computer algebra system.

In any event, we find that the length of the cardioid is  $L = 8$ .