Lecture Notes for Chapter 2-2

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MATK254 Calculus 1 Fall 2022 9-01-2022
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Calculus Early Trans Multi Term Enh Web Assign Acc

Key Topics:
The Limit of a Function

1 Intuitive Definition of a Limit

Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L \qquad \lim_{x \to a} f(x) = L$$

and say

"the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

$$f(x) = x + 3$$

$$\lim_{x \to 4} f(x) = 7$$

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = \frac{x-1}{(x+1)(x-1)} = \frac{1}{x+1}$$

$$\frac{1-1}{1^2-1} = \frac{0}{0}$$

$$\frac{x^2-1}{(x+1)(x-1)}$$

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = \frac{x-1}{(x+1)(x-1)} = \frac{1}{x+1}$$

$$= \frac{1}{1+1} = \frac{1}{2}$$

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$$

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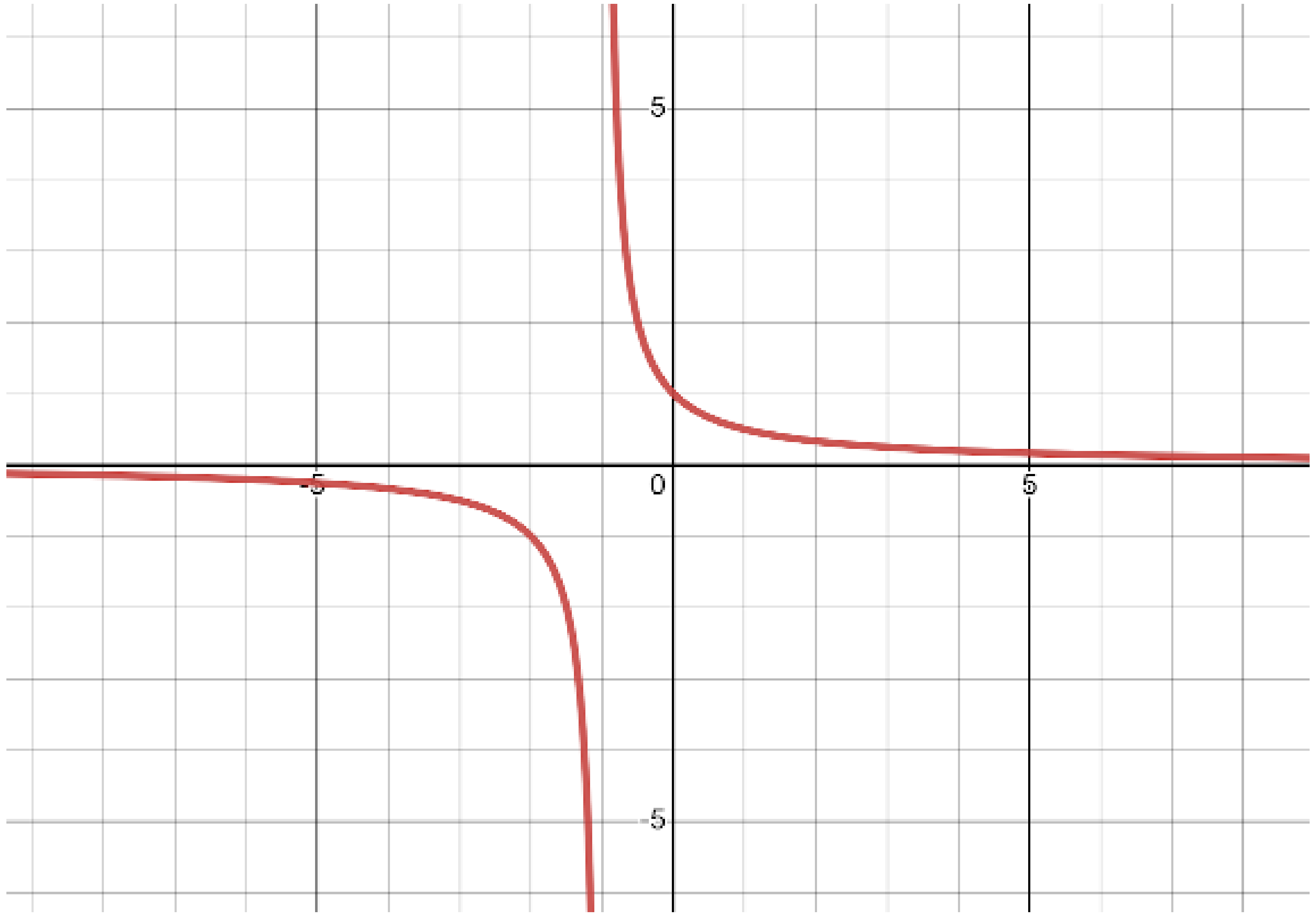


x	$\frac{x-1}{x^2-1}$
.9	0.52631579
.99	0.50251256
.999	0.50025013
.9999	0.500025
.99999	0.5000025

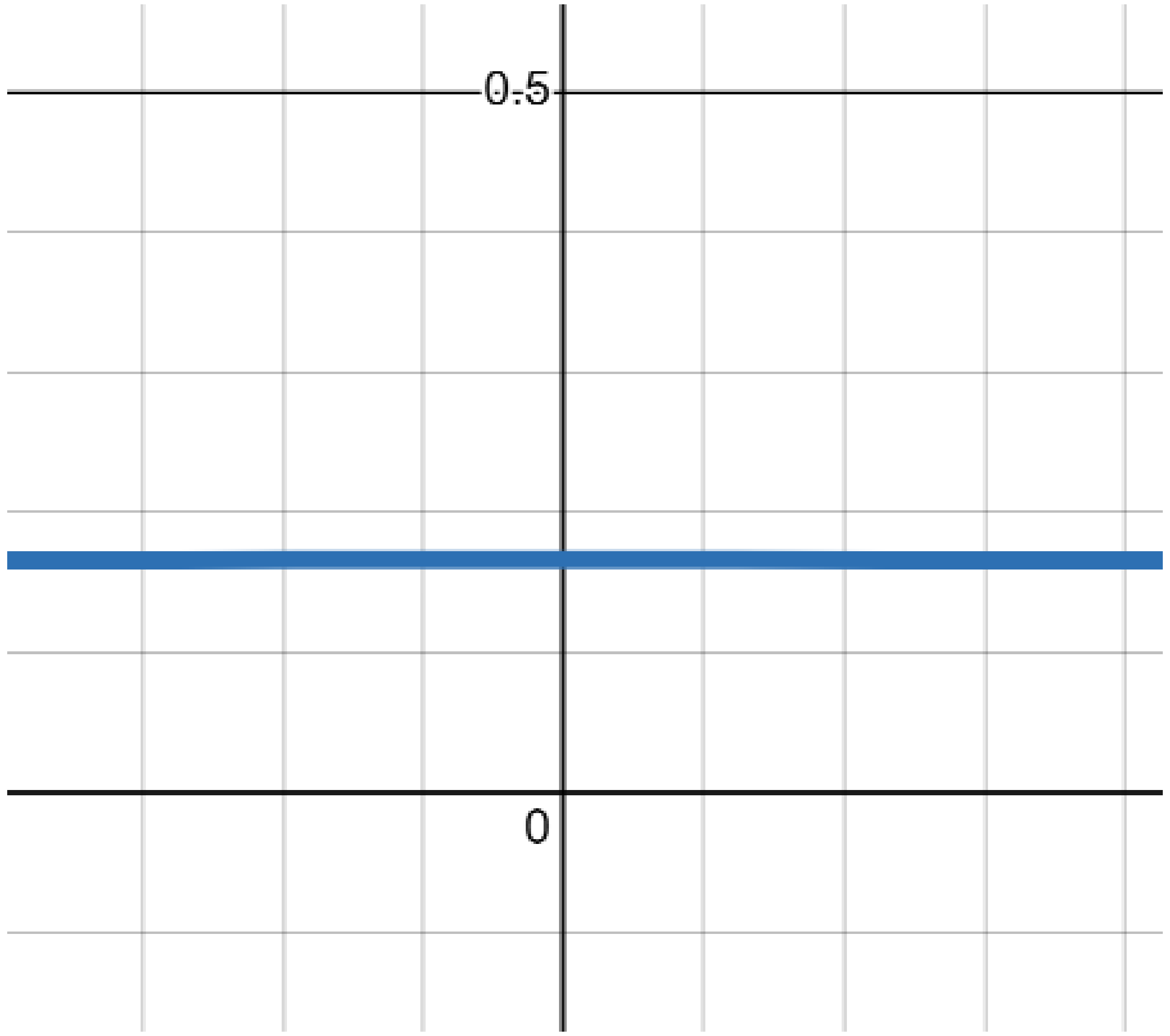
x	$\frac{x-1}{x^2-1}$
1.1	0.47619048
1.01	0.49751244
1.001	0.49975012
1.0001	0.499975
1.00001	0.4999975

Example 1.

Estimate the value of $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} \approx \frac{1}{6}$

x	$\frac{\sqrt{x^2+9}-3}{x^2}$
-1	0.16227766
$-.1$	0.1666204
$-.01$	0.1666662
$-.001$	0.16666666
$-.0001$	0.16666668

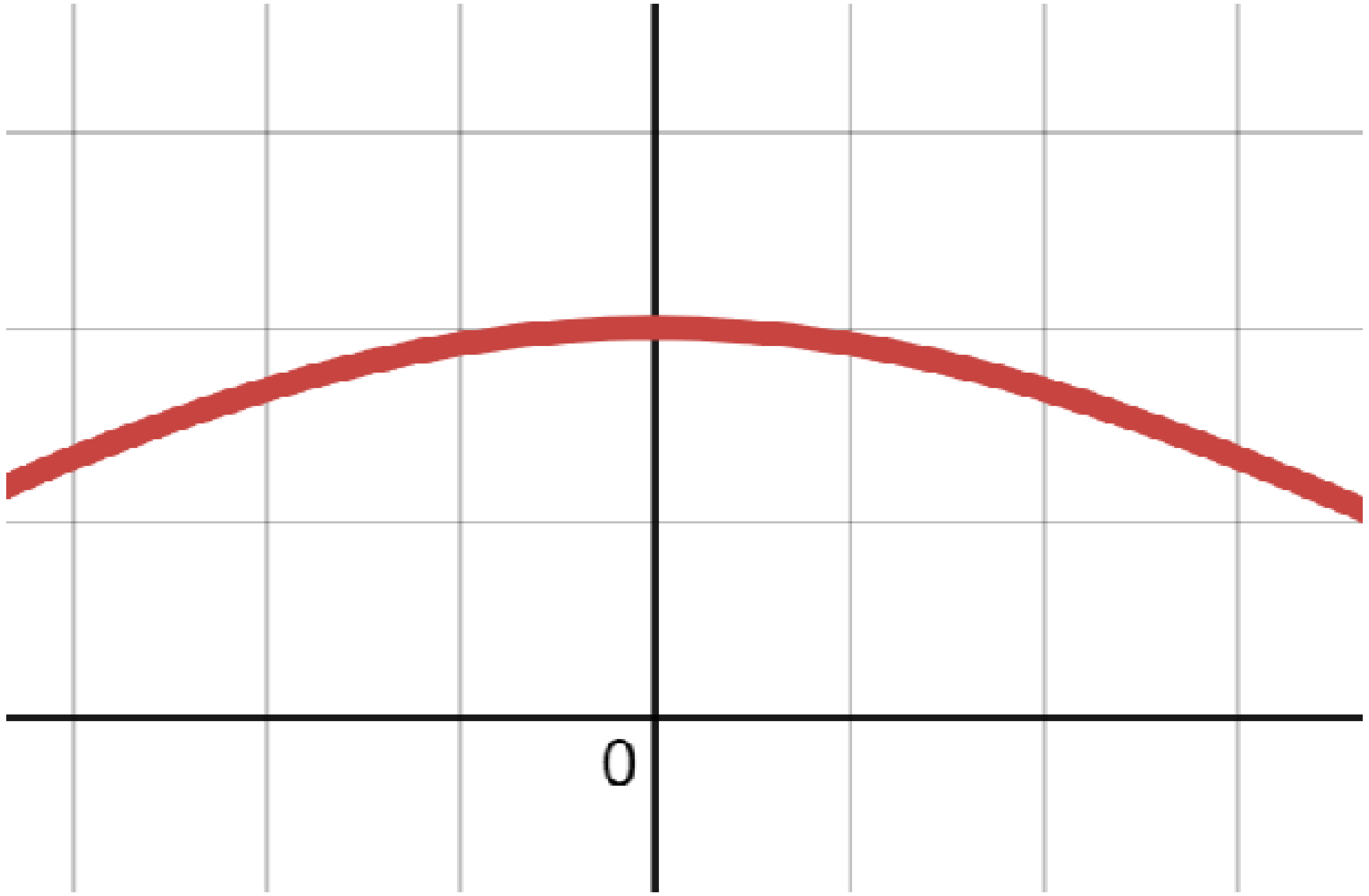
x	$\frac{\sqrt{x^2+9}-3}{x^2}$
1	0.16227766
.1	0.1666204
.01	0.1666662
.001	0.16666666
.0001	0.16666668



Example 2.

Guess the value of $\lim_{x \to 0} \frac{\sin x}{x}$.

x	$\frac{\sin x}{x}$
$-.001$	0.99999983
$.001$	0.99999983



Example 3.

Find $\lim_{x\to 0}\left(x^3 + \frac{\cos 5x}{10{,}000}\right)$ $= 0^3 + \frac{1}{10000} = \frac{1}{10000}$

2 Definition of One-Sided Limits

We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit of** $f(x)$ **as** x **approaches** a [or the **limit of** $f(x)$ **as** x **approaches** a **from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x *less than* a

$$\lim_{x \to a^-} f(x) = L \qquad \lim_{x \to a^+} f(x) = L$$

Left side

Right side

Example 7.

The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

$\lim_{x \to 2} = \text{DNE}$

(a) $\lim_{x \to 2^-} g(x)$ $\quad \lim_{x \to 2^-} = 3$

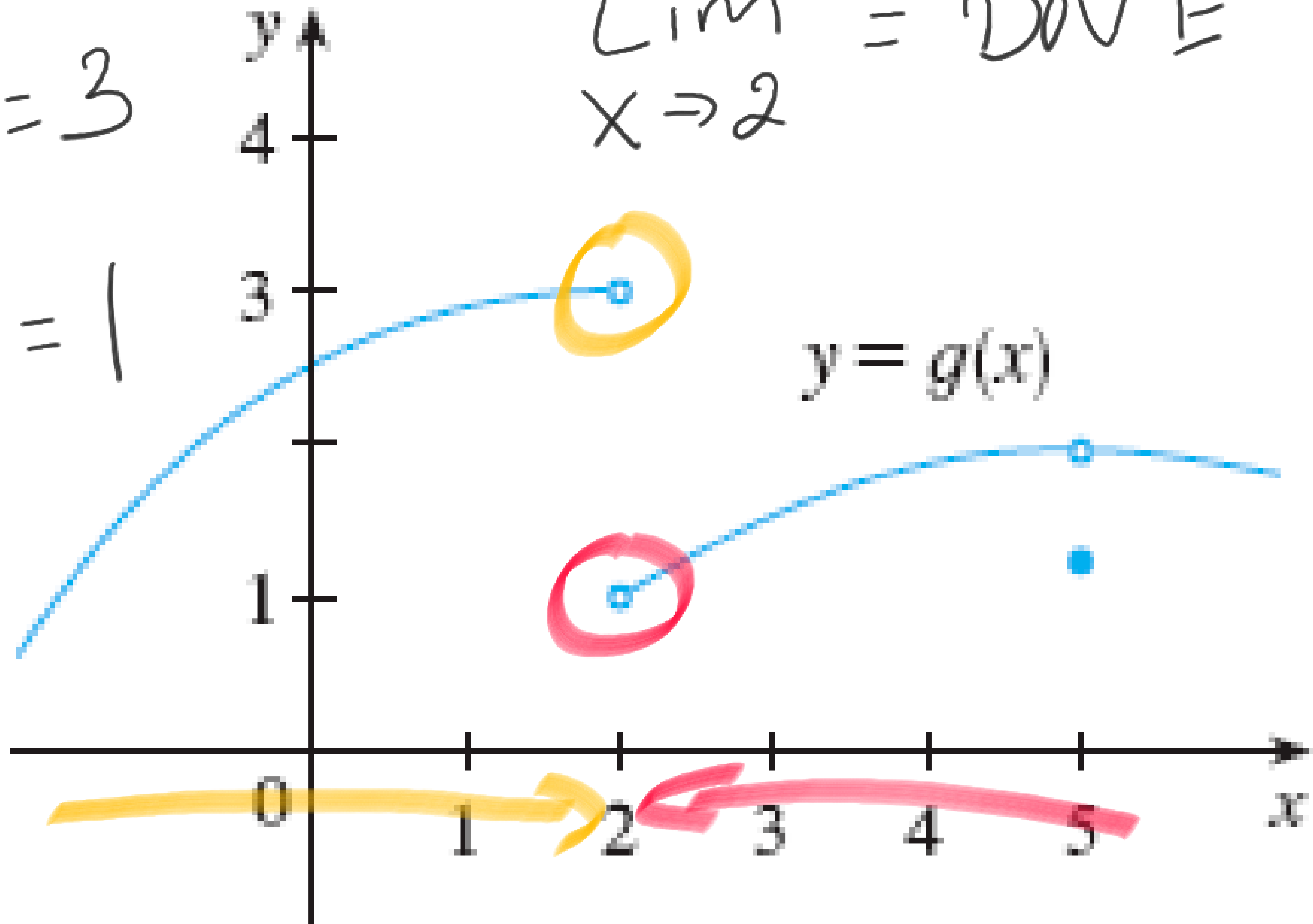
(b) $\lim_{x \to 2^+} g(x)$ $\quad \lim_{x \to 2^+} = 1$

(c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$

(e) $\lim_{x \to 5^+} g(x)$

(f) $\lim_{x \to 5} g(x)$



Example 7.

The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$

(b) $\lim_{x \to 2^+} g(x)$

(c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$

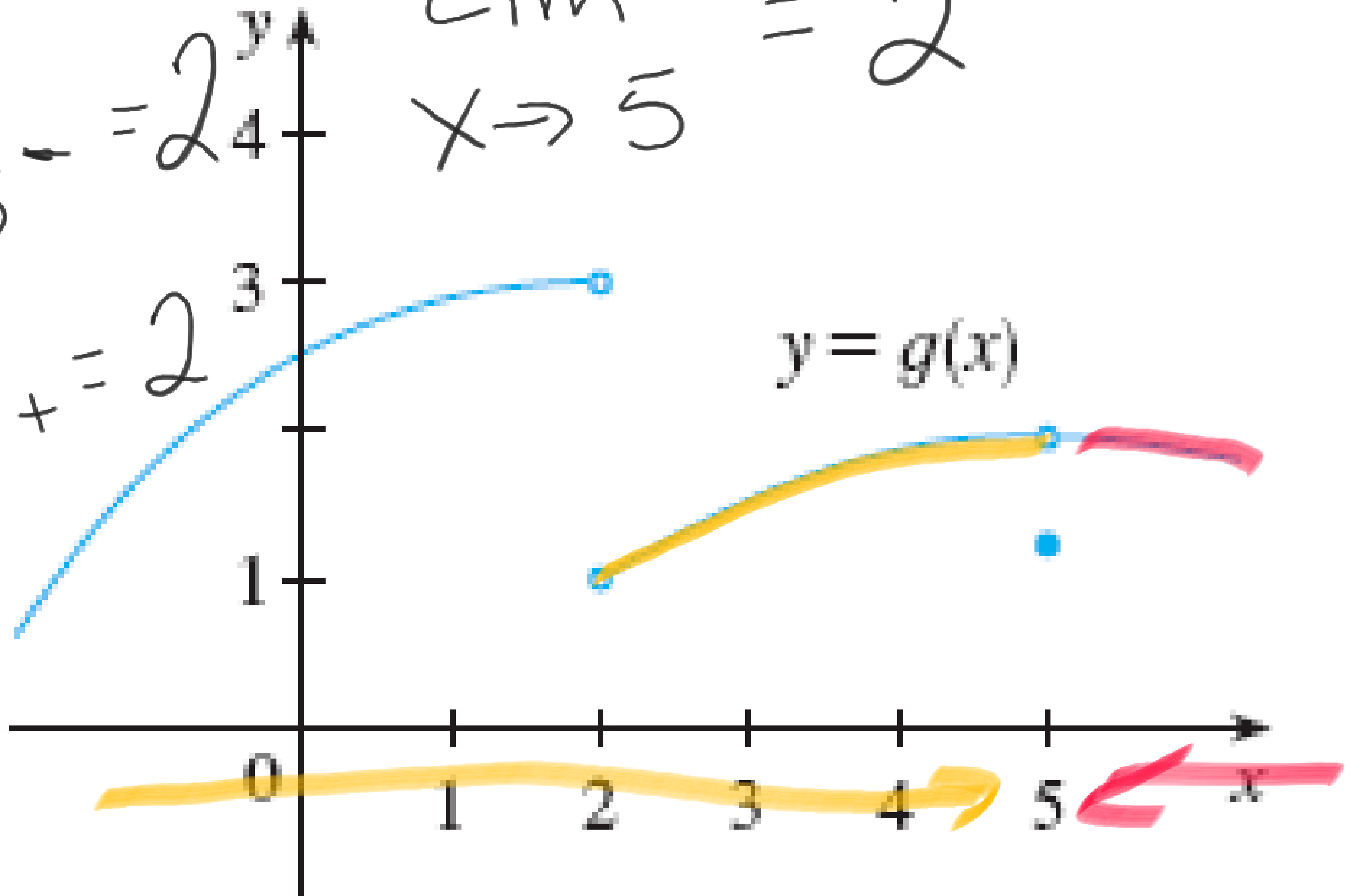
(e) $\lim_{x \to 5^+} g(x)$

(f) $\lim_{x \to 5} g(x)$

$\lim_{x \to 5^-} = 2$

$\lim_{x \to 5^+} = 2$

$\lim_{x \to 5} = 2$

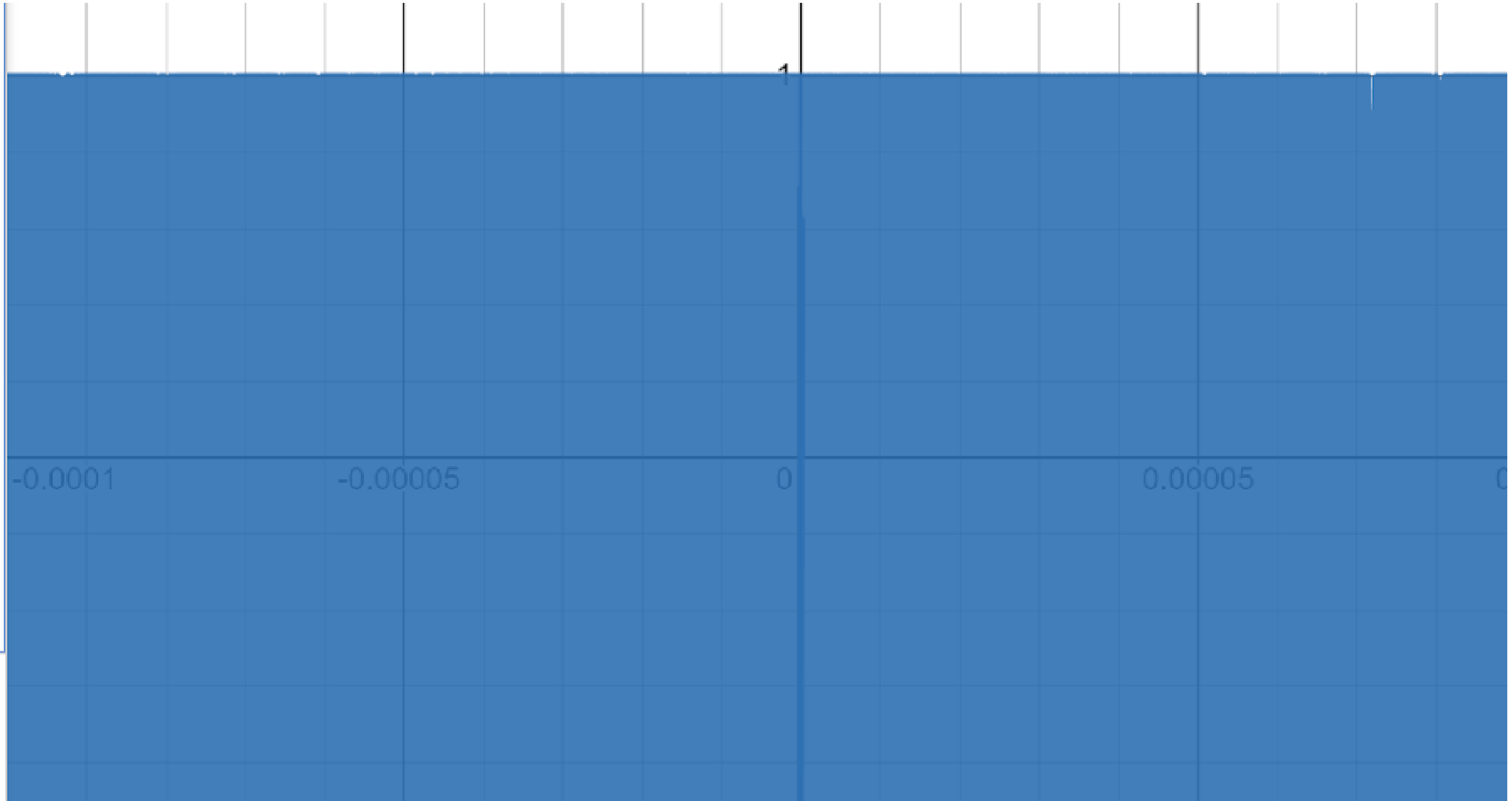


Example 5.

Investigate $\lim\limits_{x\to 0} \sin \dfrac{\pi}{x}$ = DNE

x	$\sin\left(\frac{\pi}{x}\right)$
1	0
$\frac{2}{104}$	$-2.057902 \times 10^{-14}$
$\frac{3}{1052}$	0.8660254
$\frac{4}{10327}$	-0.70710678
	undefined
.00001	0

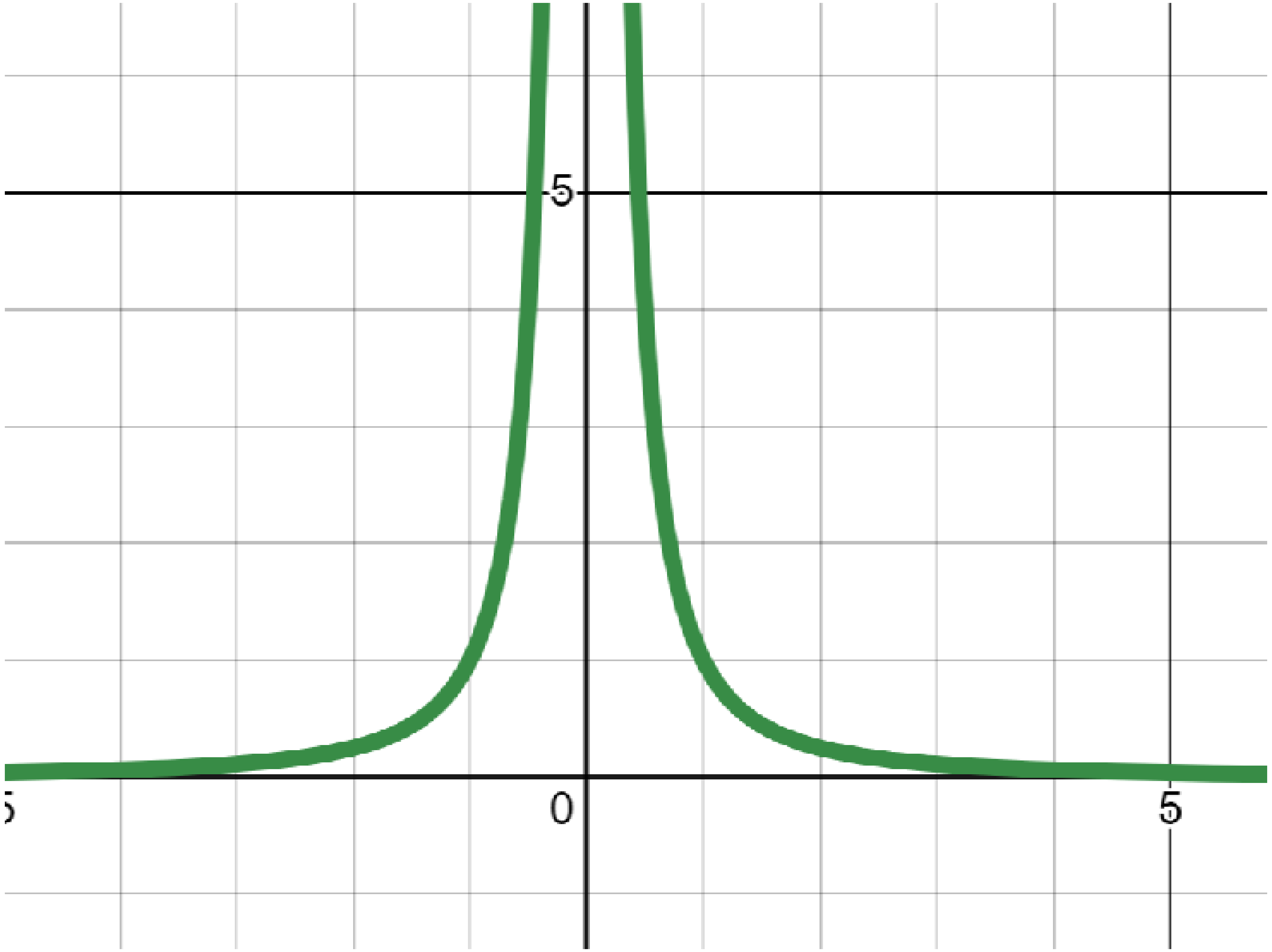
$\sin\left(\frac{\pi}{x}\right)$



Example 6.

Find $\lim\limits_{x \to 0} \dfrac{1}{x^2}$ if it exists.

$$\lim_{x \to 0} \frac{1}{x^2} \to \infty$$



4 Intuitive Definition of an Infinite Limit

Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

6 Definition

The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

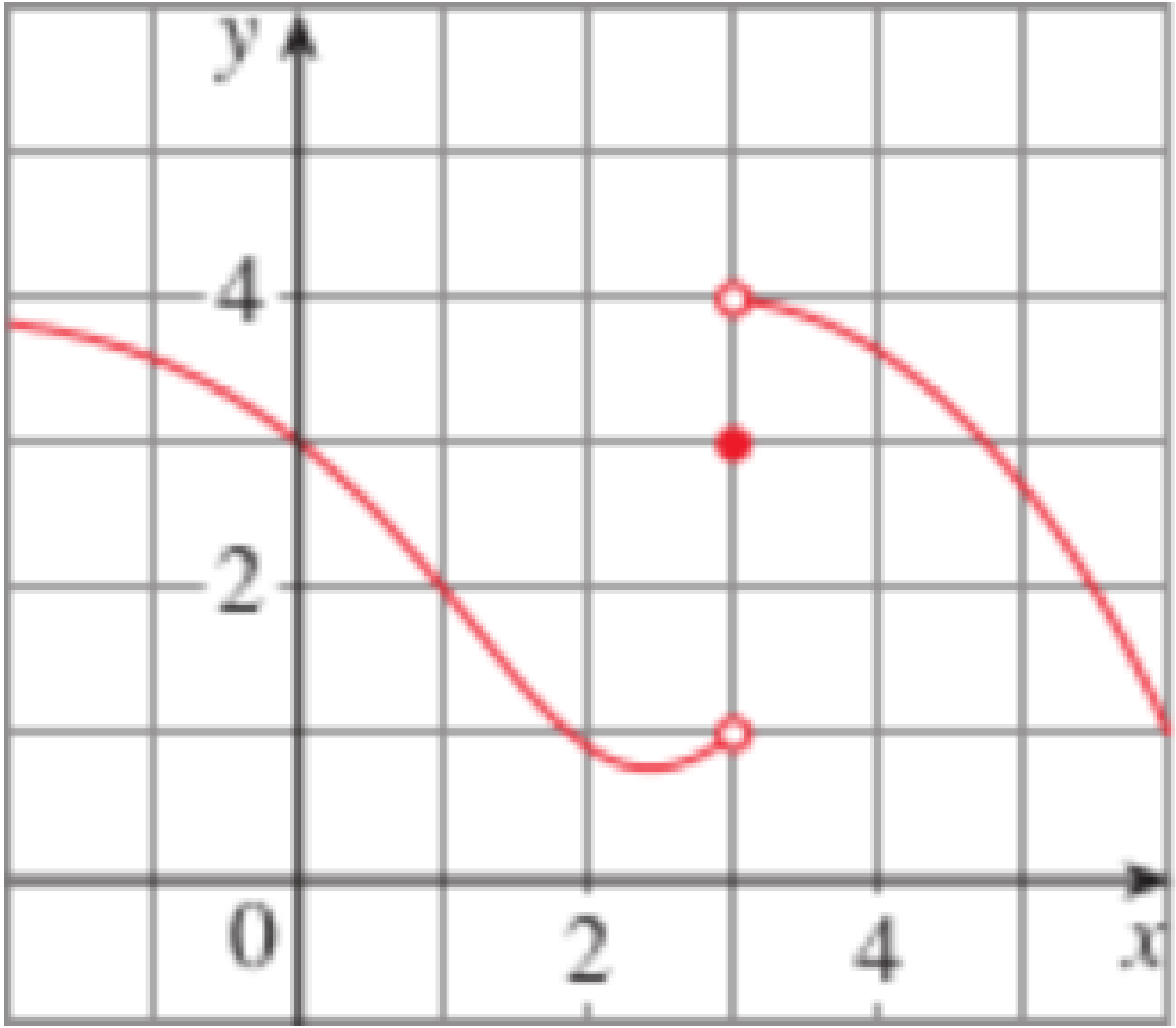
$$\lim_{x \to 0} f(x) = 3$$

$$\lim_{x \to 3^-} f(x) = 1$$

$$\lim_{x \to 3^+} f(x) = 4$$

$$\lim_{x \to 3} f(x) = DNE$$

$$f(3) = 3$$



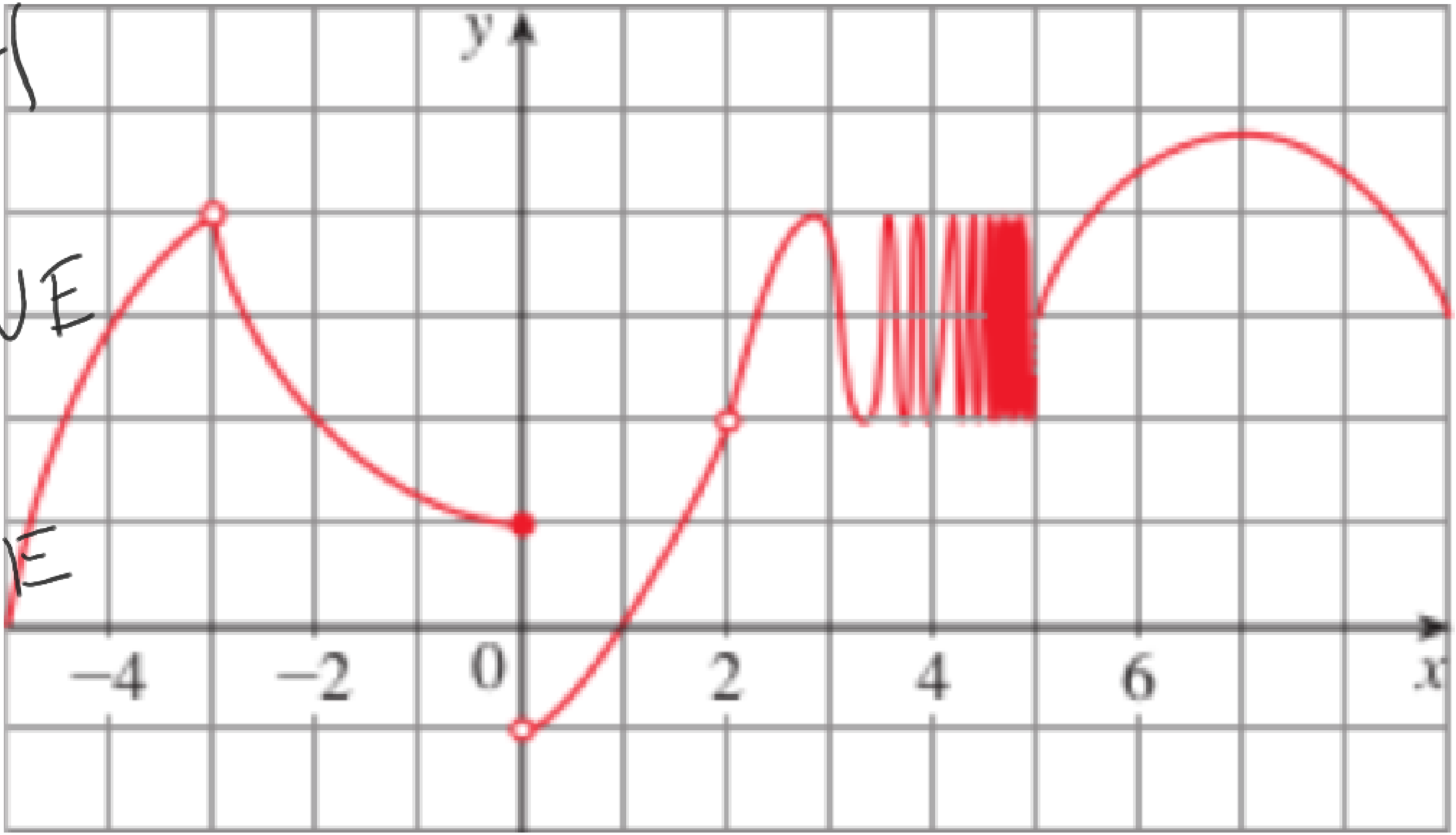
$$\lim_{x\to 2} f(x) = 2$$

$$\lim_{x\to -3} f(x) = 4$$

$$\lim_{x\to 5} f(x) = DNE$$

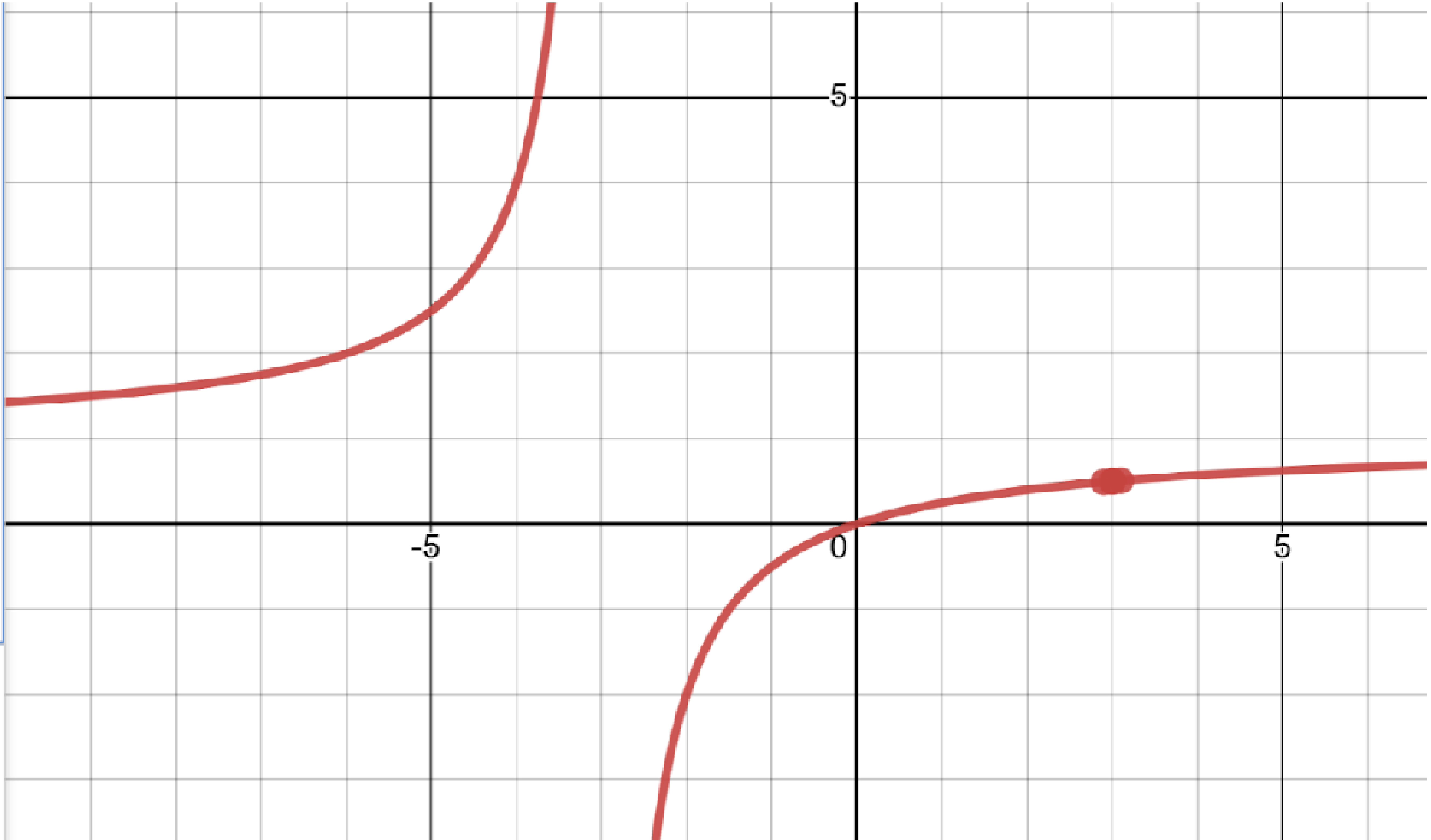
$$\lim_{x\to 5^-} f(x) = DNE$$

$$\lim_{x\to 5^+} f(x) = 3$$



$$\lim_{x\to 3} \frac{x^2-3x}{x^2-9}, = \frac{1}{2}$$

x	$\frac{x^2-3x}{x^2-9}$
2.9	0.49152542
2.99	0.49916528
2.999	0.49991665
3.001	0.50008332
3.01	0.50083195
3.1	0.50819672



$$\lim_{x\to 0}\ (1+x)^{1/x} = e$$

x	$(1+x)^{\left(\frac{1}{x}\right)}$
$-.1$	2.867972
$-.01$	2.731999
$-.001$	2.7196422
$-.0001$	2.7184178
$.0001$	2.7181459
$.001$	2.7169239
$.01$	2.7048138
$.1$	2.5937425