$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1}$$

$$\frac{x^2 - x \Rightarrow x\cancel{(x-1)}}{x^2 - 1 \Rightarrow \cancel{(x-1)}(x+1)}$$

$$\lim_{x \to 1} \frac{x}{x+1} = \frac{1}{2}$$

$$\lim_{x \to \infty} \frac{x^2/x^2 - 1/x^2}{2x^2/x^2 + 1/x^2}$$

$$\lim_{x \to \infty} \frac{1 - 1/x^2}{2 + 1/x^2} = \frac{1}{2}$$

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \frac{0}{0} \rightarrow \text{Indeterminate form}$$

$$\frac{\infty}{\infty}$$

L'Hospital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \lim_{x \to a} f(x) = \pm\infty$$

$$\lim_{x \to a} g(x) = 0 \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Find $\lim_{x \to 1} \dfrac{\ln x}{x - 1}$.

$f(x) = \ln x \quad f'(x) = 1/x$

$g(x) = x - 1 \quad g'(x) = 1$

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{1/x}{1} = 1$$

Calculate $\lim_{x\to\infty} \frac{e^x}{x^2} = \frac{\infty}{\infty}$

$f(x) = e^x \quad f'(x) = e^x \quad f''(x) = e^x$

$g(x) = x^2 \quad g'(x) = 2x \quad g''(x) = 2$

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$$

$f(x) = \ln x \qquad f'(x) = 1/x$

$g(x) = x^{1/2} \qquad g'(x) = 1/2\sqrt{x}$

$$\lim_{x \to \infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x \to \infty} \frac{1/x}{1/2\sqrt{x}} = \lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0$$

$$\frac{1}{x} \cdot \frac{2\sqrt{x}}{1} = \frac{2\sqrt{x}}{x} = \frac{2}{\sqrt{x}}$$

$$\lim_{x \to 0} \frac{\tan x - x}{x^3}$$

$f(x) = \tan x - x$ $\quad g(x) = x^3$

$f'(x) = \sec^2 x - 1$ $\quad g'(x) = 3x^2$

$f''(x)\ 2\sec^2 x \tan x$ $\quad g''(x) = 6x$

$$\lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \to 0} \frac{2\sec^2 x \tan x}{6x}$$

$$\lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} = \frac{1}{3}\lim_{x\to 0} \sec^2 x \boxed{\lim_{x\to 0} \frac{\tan x}{x}} = \frac{1}{3}$$

$$\frac{1}{3}(1) \qquad (1)$$

$f(x) = \tan x \qquad f'(x) = \sec^2 x$

$g(x) = x \qquad g'(x) = 1$

$$\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x} = \frac{0}{1-(-1)} = \frac{0}{2} = 0$$

$x \to \pi^-$

$$\lim_{x\to 0^+} x \ln x$$

$x \to 0$

$\ln x \to -\infty$

$$x = \frac{1}{\frac{1}{x}} = \frac{1}{x^{-1}}$$

$$\lim_{x\to 0^+} \frac{\ln x}{x^{-1}}$$

$f(x) = \ln x \qquad f'(x) = 1/x$

$g(x) = x^{-1} \qquad g'(x) = -x^{-2}$

$$\lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

$$\frac{1}{x} \div \frac{-1}{x^2}$$

$$\frac{1}{x} \cdot \frac{-x^2}{1} = -x$$

$$\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) = \infty - \infty$$

$$\frac{1}{\ln x} - \frac{1}{x-1} = \frac{(x-1) - \ln(x)}{(\ln x)(x-1)}$$

$$f(x) = x - 1 - \ln(x) \qquad f'(x) = 1 - 1/x$$

$$g(x) = (\ln x)(x-1) \qquad g'(x) = \left(\frac{1}{x}\right)(x-1) + (\ln x)(1)$$

$$= \frac{x-1}{x} + \ln x$$

$$f'(x) = 1 - 1/x$$

$$f'(x) = \frac{1}{1} - \frac{1}{x} = \frac{(x)-(1)}{x}$$

$$g'(x) = \left(\frac{1}{x}\right)(x-1) + (\ln x)(1)$$

$$= \frac{x-1}{x} + \ln x$$

$$g'(x) = \frac{x-1}{x} + \frac{\ln x}{1} = \frac{(x-1)+(x\ln x)}{x}$$

$$\frac{x-1}{x} \div \frac{x-1+x\ln x}{x} = \frac{x-1}{x} \cdot \frac{x}{x-1+x\ln x}$$

$$= \frac{x-1}{x-1+x\ln x}$$

$$\lim_{x\to 1^+} \frac{x-1}{x-1+x\ln x}$$

$f'(x) = x-1$ $\qquad f''(x) = 1$

$g'(x) = x-1+x\ln x$ $\qquad g''(x) = 1 + \ln x + x\left(\frac{1}{x}\right)$ $\quad 1$

$= 2+\ln x$

$$\lim_{x\to 1^+} \frac{1}{2+\ln x} = \boxed{\frac{1}{2}}$$

$$\lim_{x \to 0^+} (1+\sin 4x)^{\cot x}$$

$0^0,\ \infty^0,\ 1^\infty$

$$y = (1+\sin 4x)^{\cot x}$$

$$\ln y = \cot x \ln(1+\sin 4x)$$

$$\ln y = \frac{\ln(1+\sin 4x)}{\tan x}$$

$$\ln y = \frac{\ln(1+\sin 4x)}{\tan x}$$

$$f(x) = \ln(1+\sin 4x) \qquad f'(x) = \frac{4\cos 4x}{1+\sin 4x}$$

$$g(x) = \tan x \qquad g'(x) = \sec^2 x$$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\frac{4\cos 4x}{1+\sin 4x}}{\sec^2 x} = \frac{\frac{4}{1}}{1} = 4$$

$$\lim_{x \to 0^+} e^{\ln y} = e^4$$

$$y = e^{\ln y}$$

$$\lim_{x \to 0} e^{\ln y} = e^4$$

$$\boxed{y = e^4}$$

$$\lim_{x \to 0^+} e^{\ln y} = e^4$$

$$\lim_{x \to 0^+} x^x$$

$$\boxed{x = e^{\ln x}}$$

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0} \left(e^{\ln x}\right)^x$$

$$= \lim_{x \to 0} e^{\boxed{x \ln x}} = e^0 = \boxed{1}$$

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