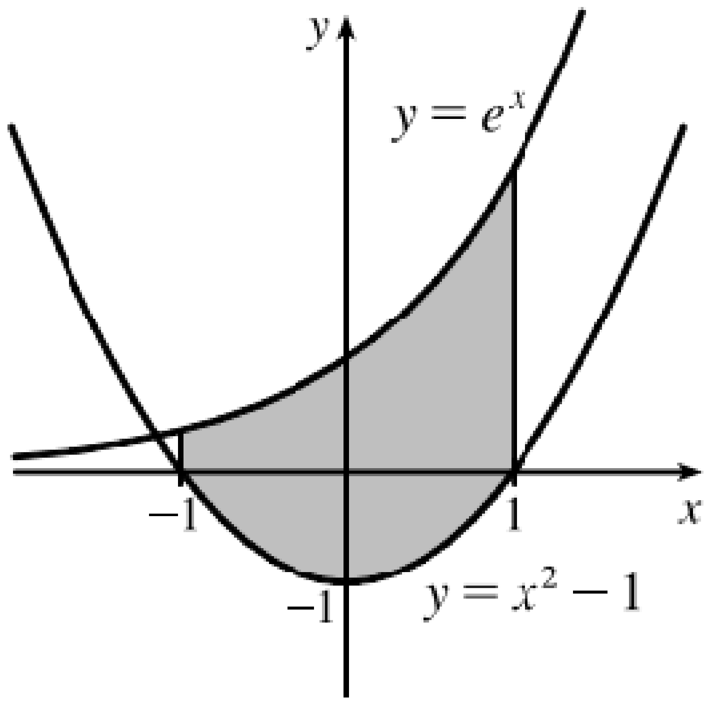
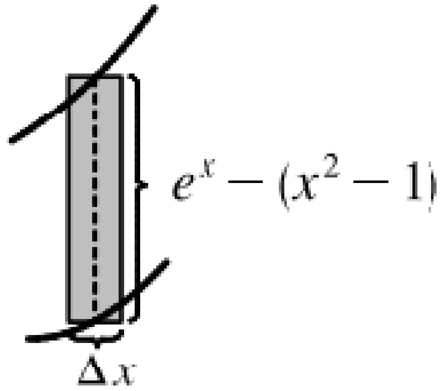
5. $A = \int_{-1}^{1} \left[e^x - (x^2 - 1)\right] dx = \left[e^x - \frac{1}{3}x^3 + x\right]_{-1}^{1}$

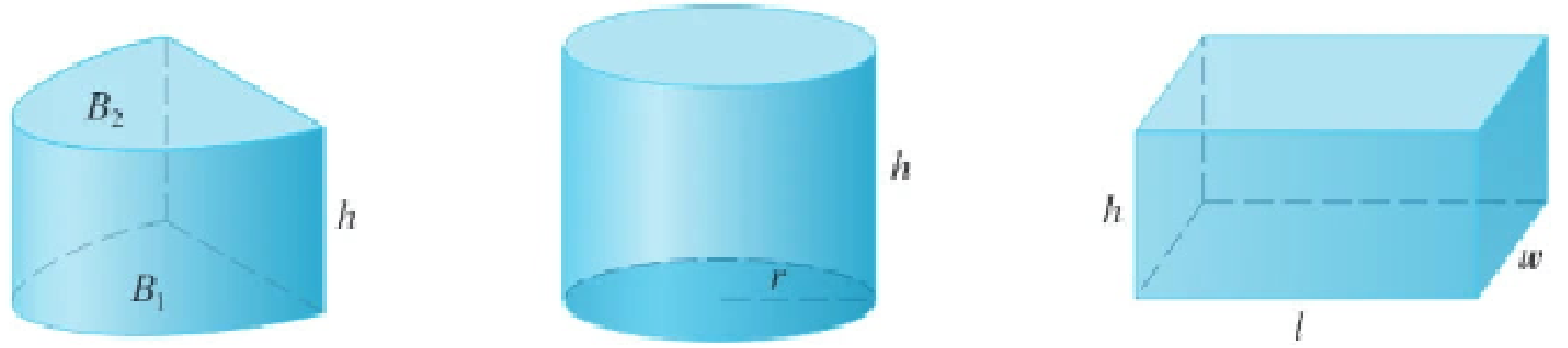
$= (e - \frac{1}{3} + 1) - (e^{-1} + \frac{1}{3} - 1) = e - \frac{1}{e} + \frac{4}{3}$



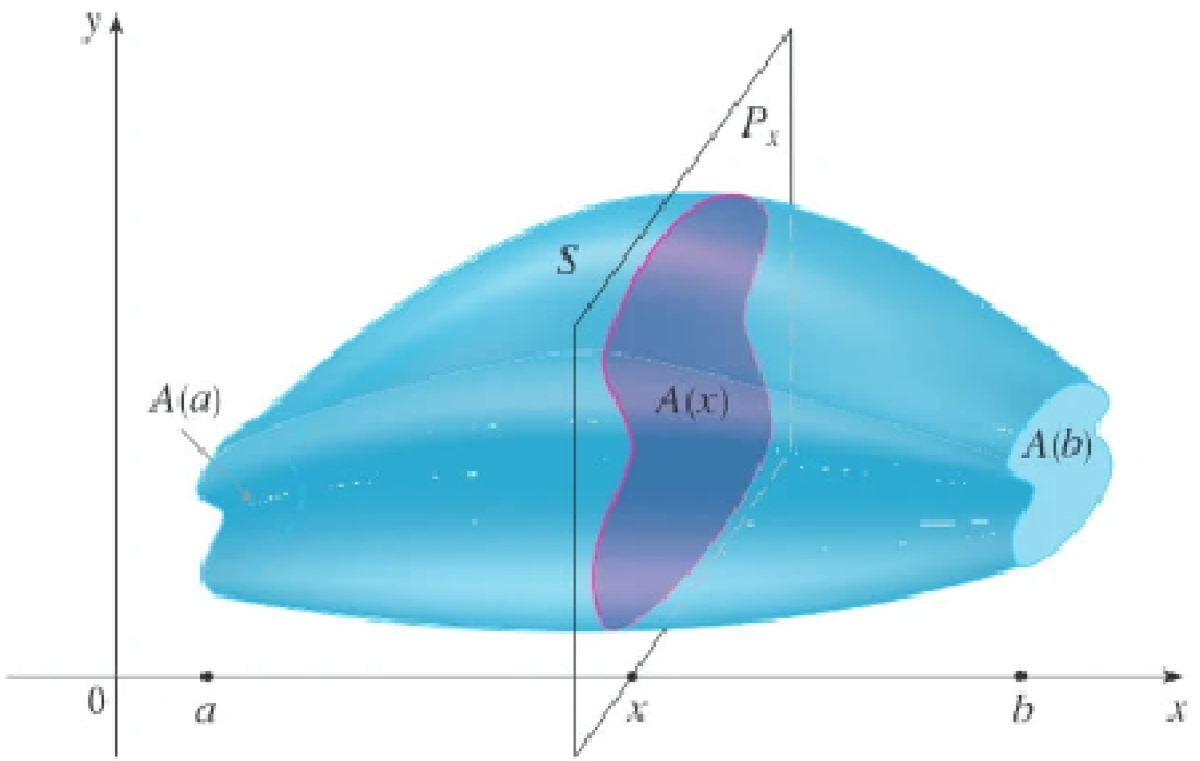


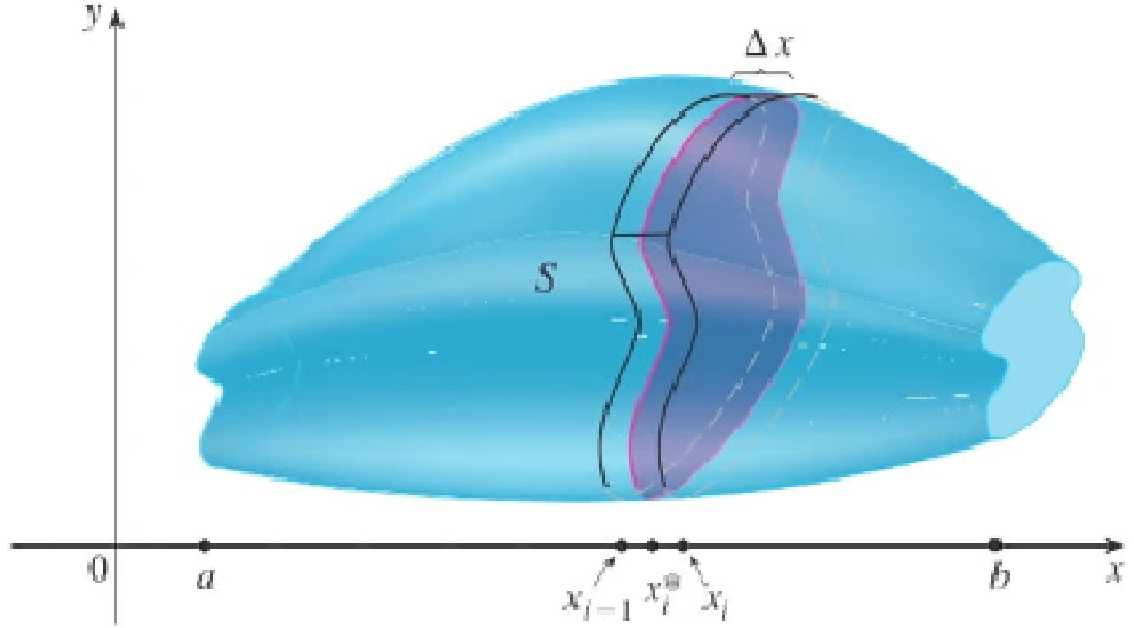
$$V = Ah$$

In particular, if the base is a circle with radius r, then the cylinder is a circular cylinder with volume $V = \pi r^2 h$ [see Figure 1(b)], and if the base is a rectangle with length l and width w, then the cylinder is a rectangular box (also called a *rectangular parallelepiped*) with volume $V = lwh$ [see Figure 1(c)].



(a) Cylinder $V = Ah$ (b) Circular cylinder $V = \pi r^2 h$ (c) Rectangular box $V = lwh$

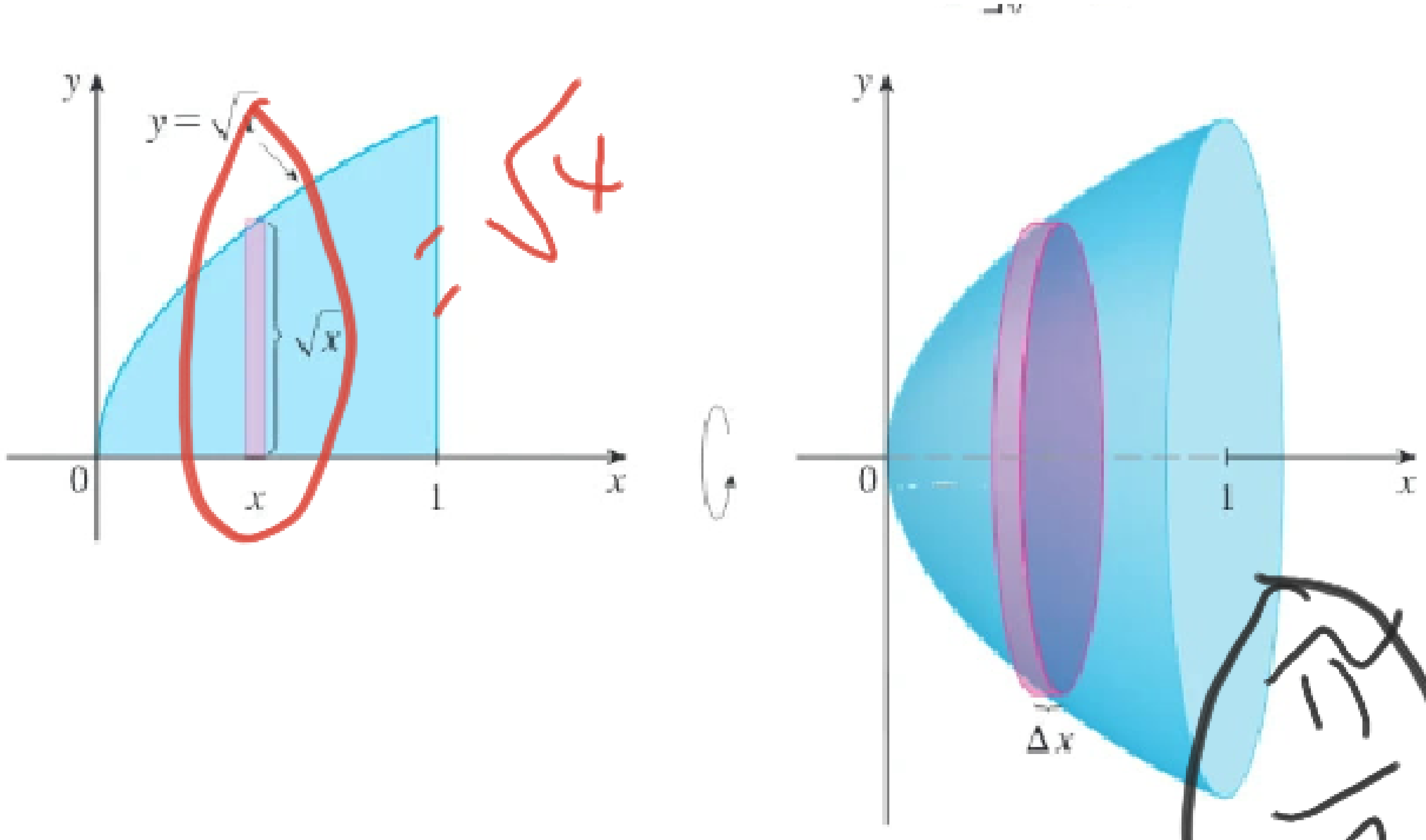




Definition of Volume Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$

EXAMPLE 2 Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. Illustrate the definition of volume by sketching a typical approximating cylinder.



(a) (b)

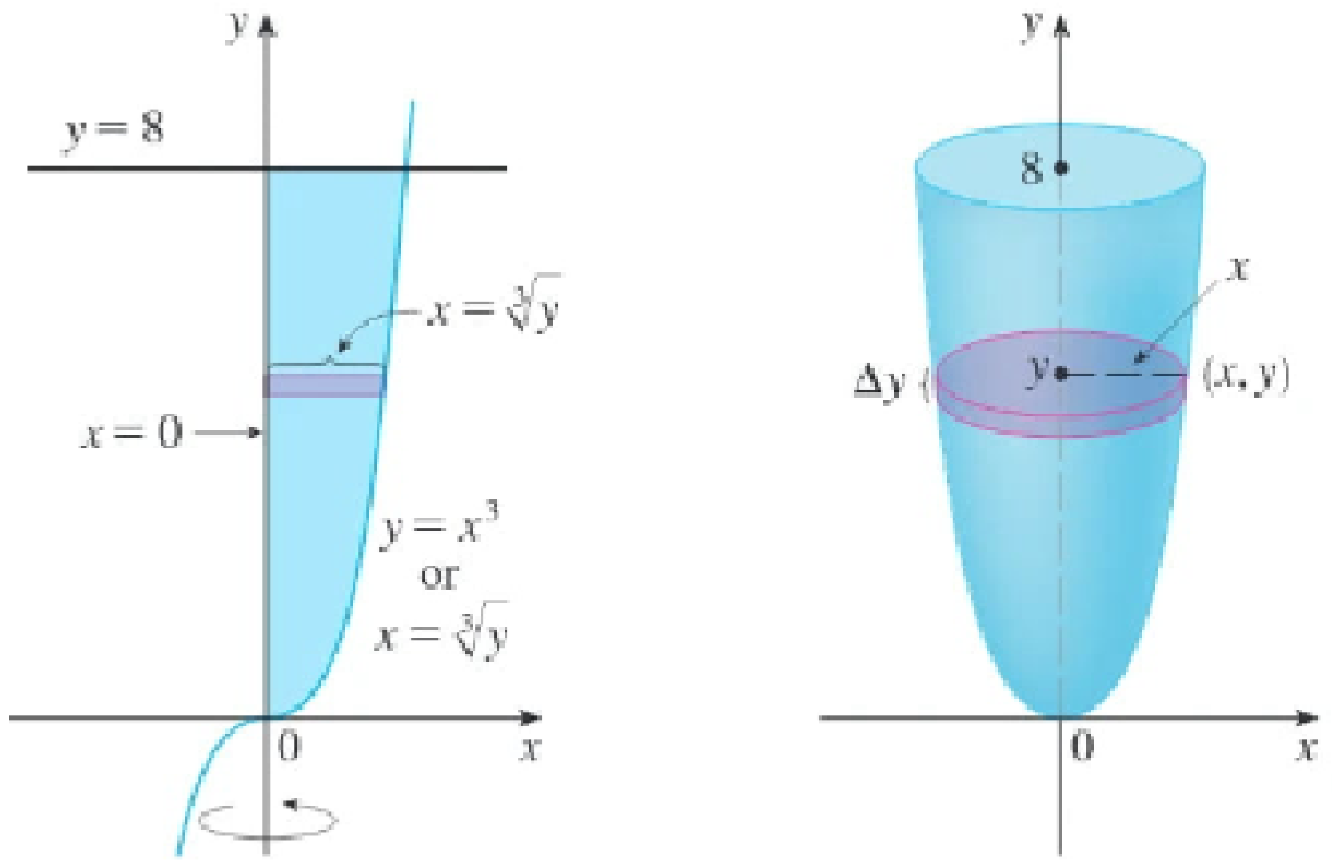
$y = \sqrt{x}$ $\qquad A = \pi r^2$

$A(x) = \pi(\sqrt{x})^2$

$= \pi x$

$\frac{\pi}{2} = \int_0^1 \pi x \, dx = \frac{\pi x^2}{2} \Big]_0^1$

EXAMPLE 3 Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.



(a) (b)

$y = x^3 \quad y = 8 \quad y = 0$

y-axis

$y = x^3 \to \sqrt[3]{y} = x$

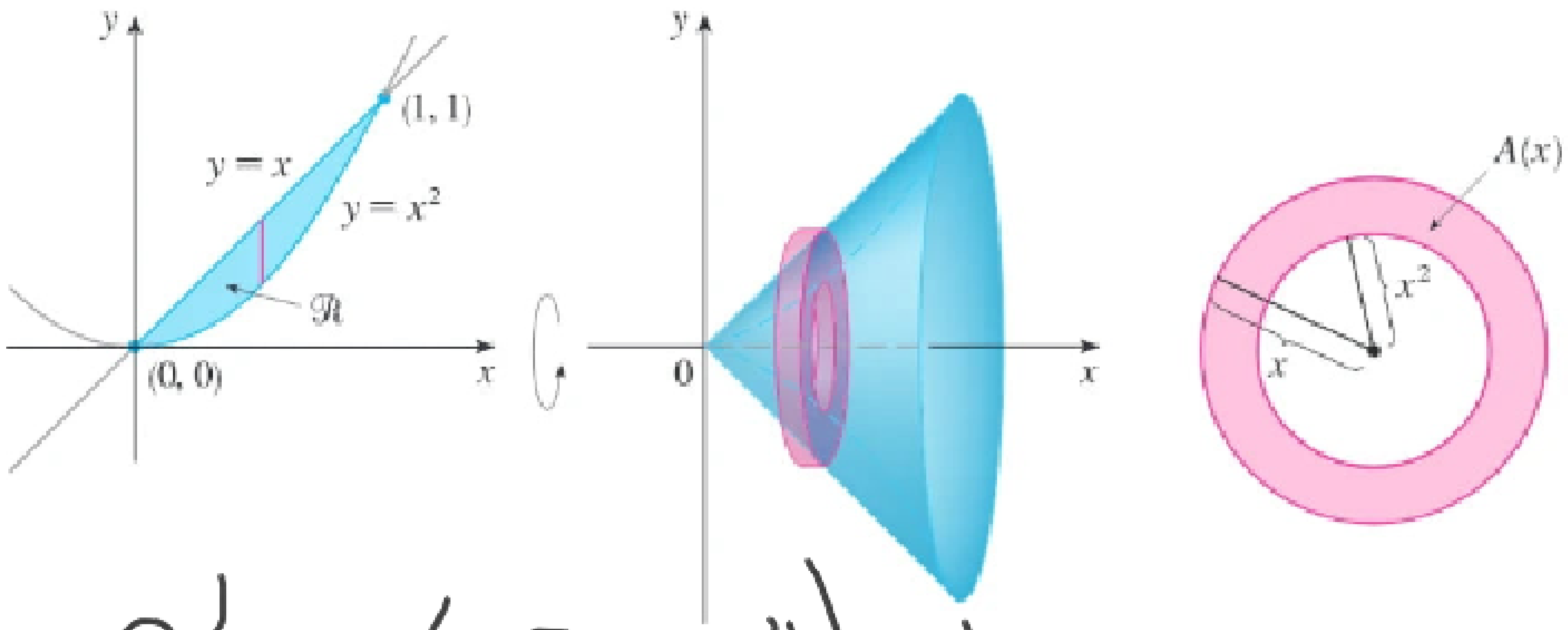
$$A(y) = \pi\left(\sqrt[3]{y}\right)^2$$

$$A(y) = \pi y^{2/3}$$

$$\int_0^8 \pi y^{2/3}\,dy = \pi\left(\frac{3y^{5/3}}{5}\right)\Bigg]_0^8$$

$$\pi \frac{96}{5} = \frac{96\pi}{5}$$

EXAMPLE 4 The region $\mathscr{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.



$y = x \quad y = x^2$

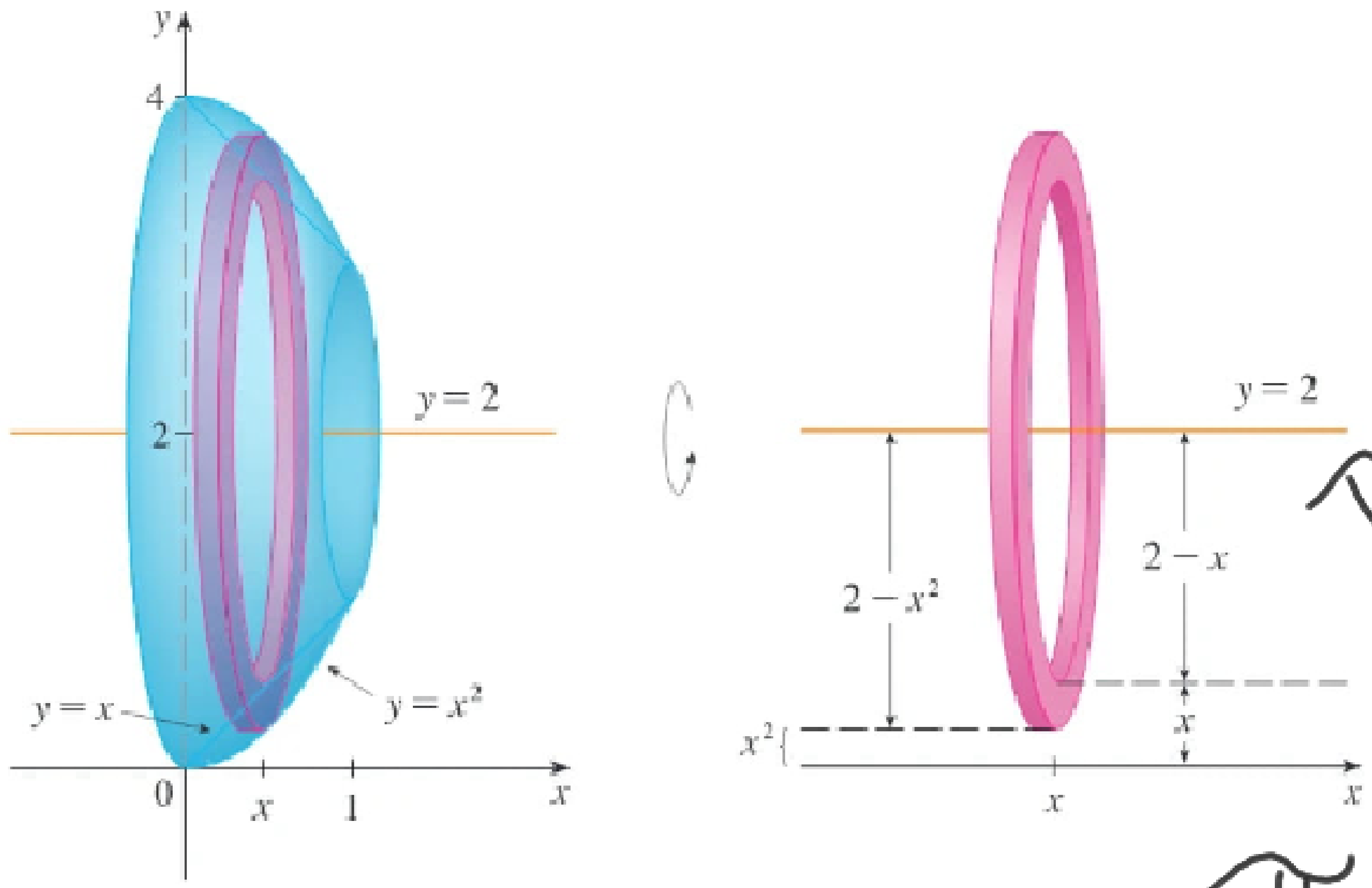
$A(x) = \pi(x)^2 - \pi(x^2)^2$

$A(x) = \pi(x^2 - x^4)$

$\int_0^1 \pi(x^2 - x^4)\,dx$

$\pi\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$

EXAMPLE 5 Find the volume of the solid obtained by rotating the region in Example 4 about the line $y = 2$.



$$A(x) = \pi\left(x^4 - 5x^2 + 4x\right)$$

$$\pi \int_0^1 x^4 - 5x^2 + 4x \, dx$$

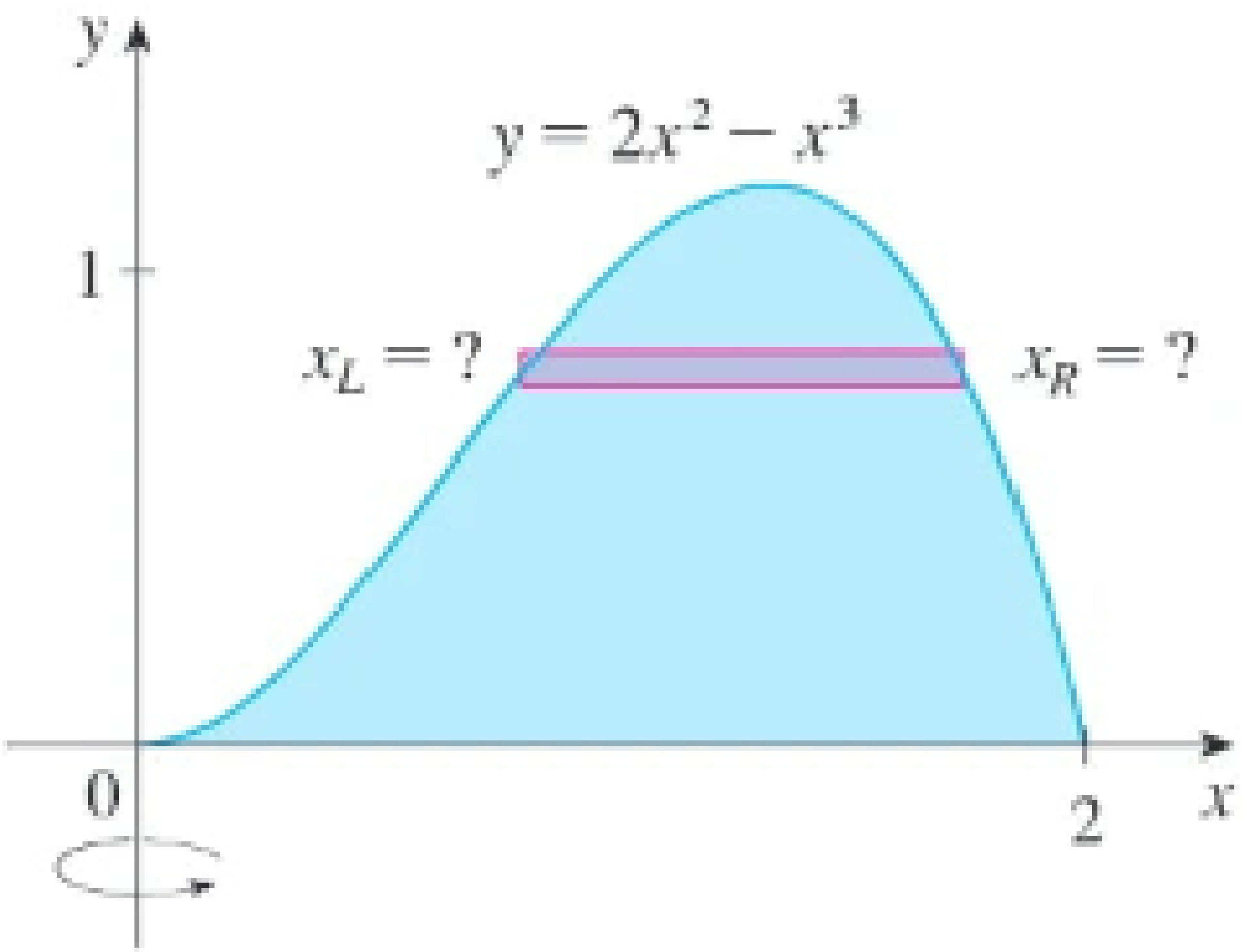
$$\pi\left[\frac{x^5}{5} - \frac{5x^3}{3} + \frac{4x^2}{2}\right]_0^1$$

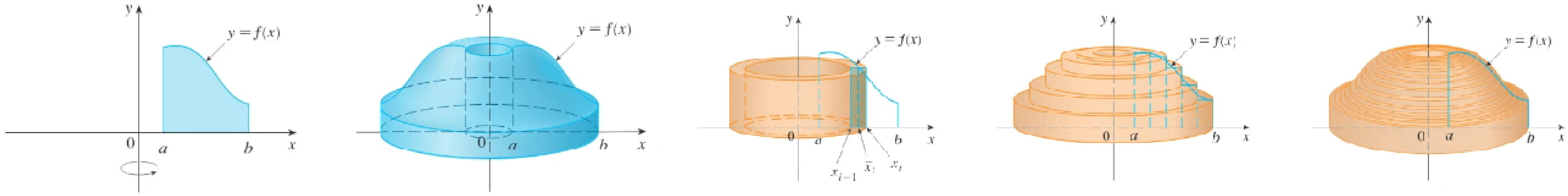
$$\pi\left(\frac{1}{5} - \frac{5}{3} + 2\right) = \frac{8\pi}{15}$$

$$A(\text{Blue}) = \pi(2-x)^2$$

$$A(\text{purple}) = \pi(2-x^2)^2$$

$$A(x) = \pi(2-x^2)^2 - \pi(2-x)^2$$


y
$y = 2x^2 - x^3$
1
$x_L = ?$
$x_R = ?$
0
2
x



RE 3

2 The volume of the solid in Figure 3, obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b, is

$$V = \int_a^b 2\pi x f(x)\, dx \qquad \text{where } 0 \leqslant a < b$$

$$V = 2\pi \int_a^b x f(x)\, dx$$

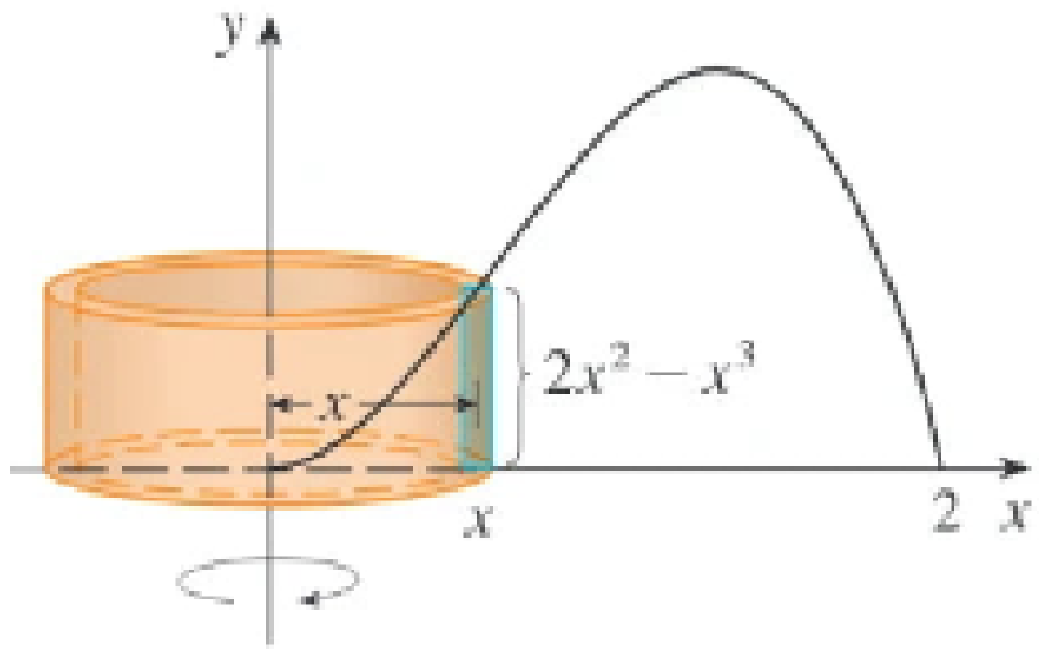
$$A(x) = \pi(2-x^2)^2 - \pi(2-x)^2$$

$$-4 + 4x - x^2$$

$$= \pi\left[(4 - 4x^2 + x^4) - (4 - 4x + x^2)\right]$$

$$= \pi(x^4 - 5x^2 + 4x)$$

EXAMPLE 1 Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.



$y = 2x^2 - x^3$ $\quad 0 \to 2$

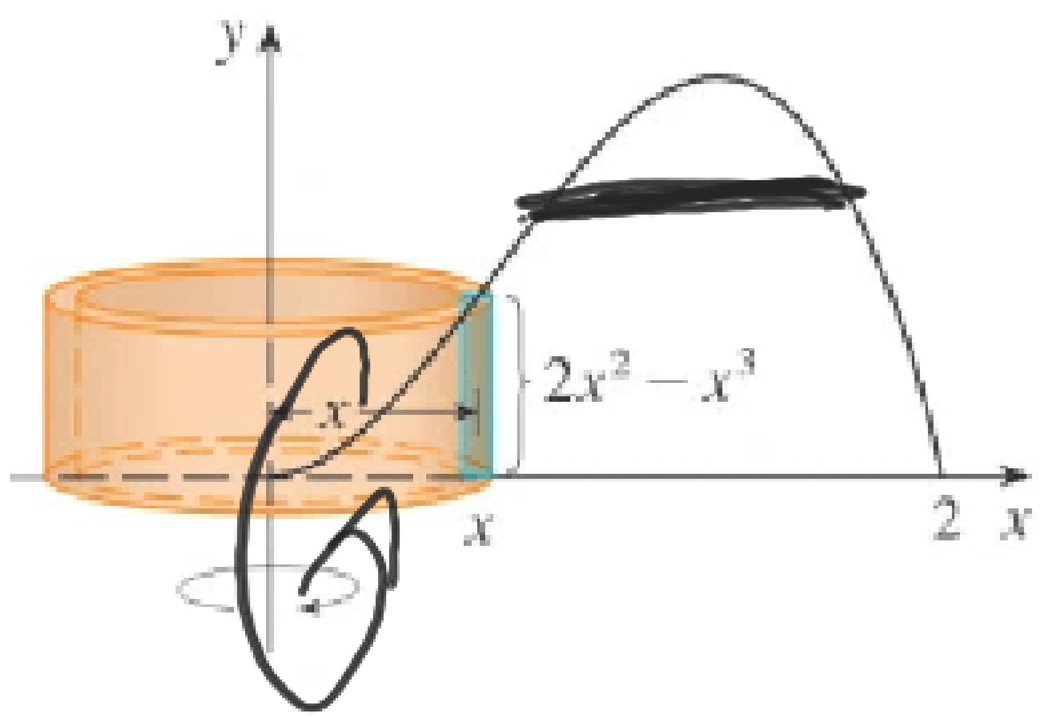
$$2\pi \int_0^2 x\left(2x^2 - x^3\right)dx$$

$$2x^3 - x^4$$

$$2\pi \left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2$$

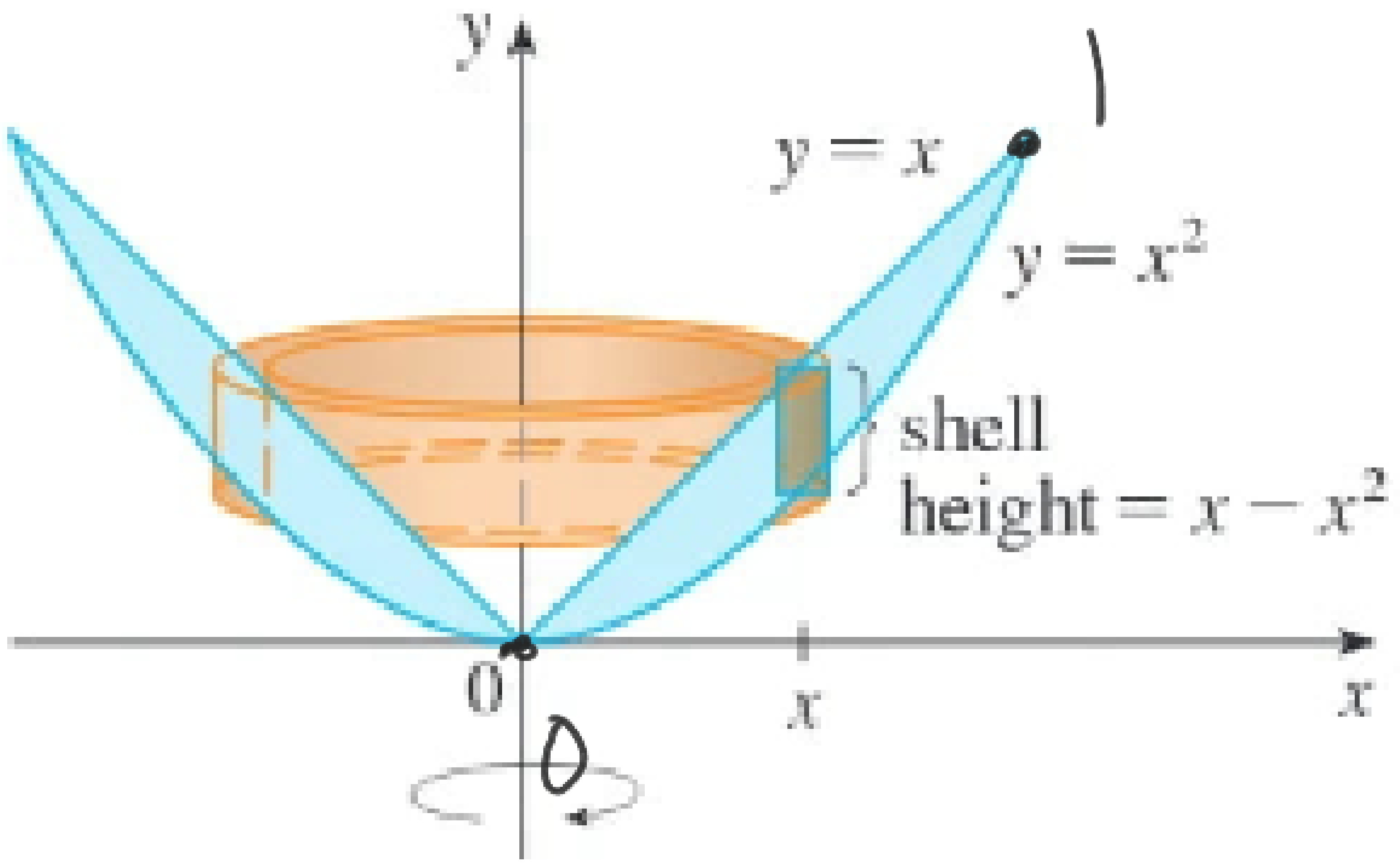
$$2\pi \left[8 - \frac{32}{5}\right) = \frac{16\pi}{5}$$

EXAMPLE 1 Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.



$$y = 2x^2 - x^3$$

EXAMPLE 2 Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.



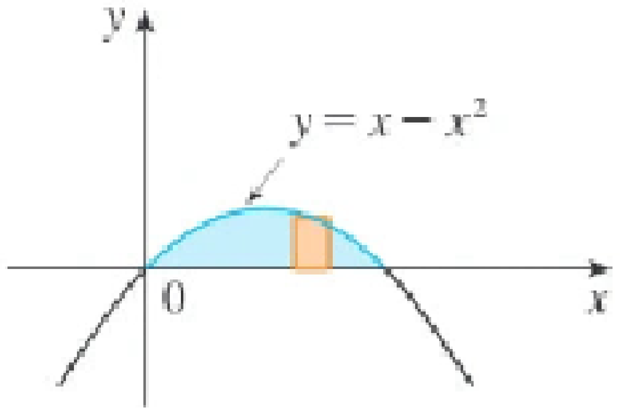
$$V = \int_0^1 2\pi x(x - x^2)\,dx$$

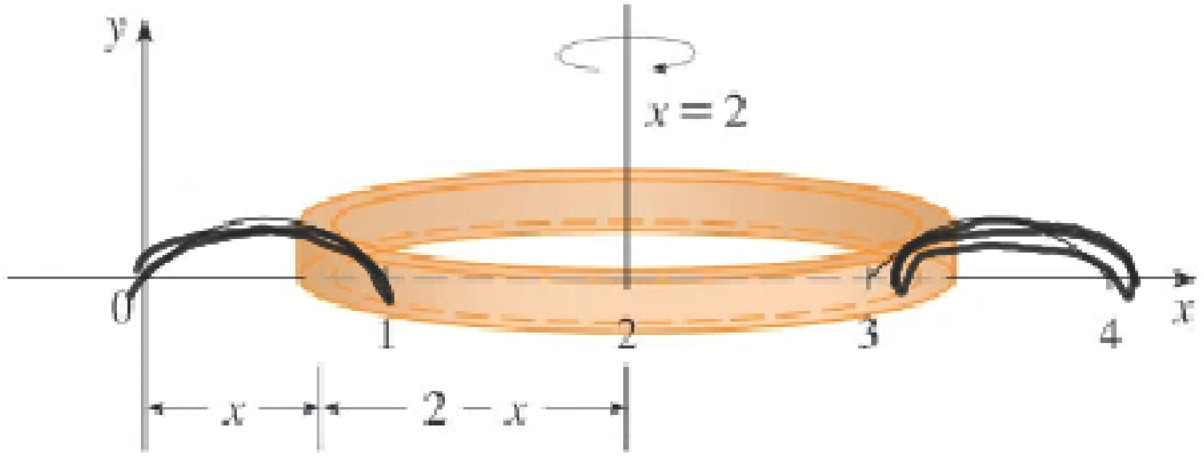
$$V = 2\pi \int_0^1 x^2 - x^3\,dx$$

$$2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$\boxed{\frac{\pi}{6}}$$

EXAMPLE 4 Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.





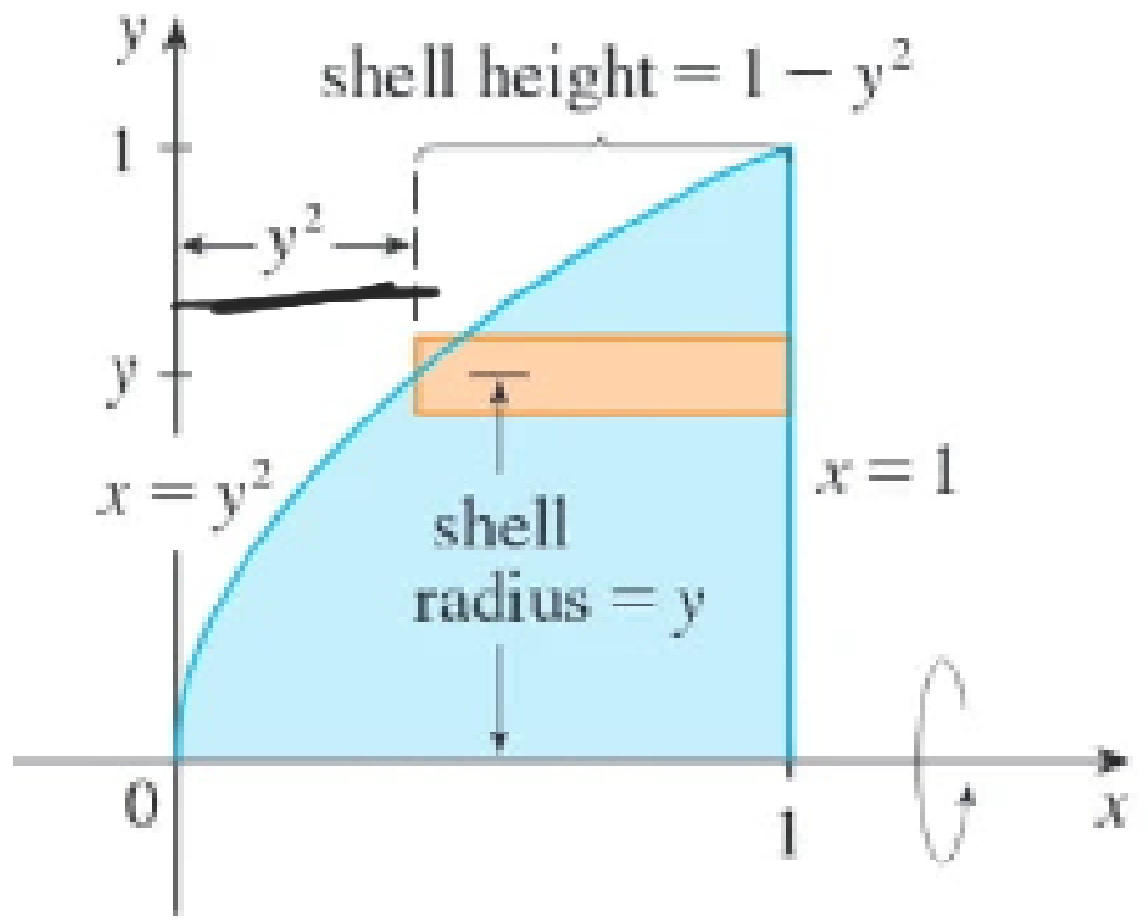
$2x - 2x^2 - x^2 + x^3$

$$V = \int_0^1 2\pi (2 - x)(x - x^2)$$

$$V = 2\pi \int_0^1 x^3 - 3x^2 + 2x \quad dx$$

$$V = 2\pi \left[\frac{x^4}{4} - \frac{3x^3}{3} + \frac{2x^2}{2} \right]_0^1 = 2\pi \left(\frac{1}{4} - 1 + 1 \right) = \frac{\pi}{2}$$

EXAMPLE 3 Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.



$$y = \sqrt{x} \rightarrow x = y^2$$

$$\int_0^1 2\pi y(1-y^2)\,dy$$

$$2\pi \int_0^1 y - y^3\,dy$$

$$2\pi \left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\pi}{2}$$