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## Applications of Integration



6.4

# Work

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# Work

The term *work* is used in everyday language to mean the total amount of effort required to perform a task.

In physics it has a technical meaning that depends on the idea of a *force*.

Intuitively, you can think of a force as describing a push or pull on an object—for example, a horizontal push of a book across a table or the downward pull of the earth's gravity on a ball.

# Work

In general, if an object moves along a straight line with position function  $s(t)$ , then the **force**  $F$  on the object (in the same direction) is given by Newton's Second Law of Motion as the product of its mass  $m$  and its acceleration  $a$ :

$$\boxed{1} \quad F = m \frac{d^2s}{dt^2}$$

In the SI metric system, the mass is measured in kilograms (kg), the displacement in meters (m), the time in seconds (s), and the force in newtons ( $\text{N} = \text{kg} \cdot \text{m}/\text{s}^2$ ). Thus a force of 1 N acting on a mass of 1 kg produces an acceleration of  $1 \text{ m}/\text{s}^2$ .

# Work

In the US Customary system the fundamental unit is chosen to be the unit of force, which is the pound.

In the case of constant acceleration, the force  $F$  is also constant and the work done is defined to be the product of the force  $F$  and the distance  $d$  that the object moves:

$$\boxed{2} \quad W = Fd \quad \text{work} = \text{force} \times \text{distance}$$

# Work

If  $F$  is measured in newtons and  $d$  in meters, then the unit for  $W$  is a newton-meter, which is called a joule (J).

If  $F$  is measured in pounds and  $d$  in feet, then the unit for  $W$  is a foot-pound (ft-lb), which is about 1.36 J.

# Example 1

- (a) How much work is done in lifting a 1.2-kg book off the floor to put it on a desk that is 0.7 m high? Use the fact that the acceleration due to gravity is  $g = 9.8 \text{ m/s}^2$ .
- (b) How much work is done in lifting a 20-lb weight 6 ft off the ground?

# Example 1(a) – *Solution*

The force exerted is equal and opposite to that exerted by gravity, so Equation 1 gives

$$\begin{aligned} F &= mg = (1.2)(9.8) \\ &= 11.76 \text{ N} \end{aligned}$$

and then Equation 2 gives the work done as

$$\begin{aligned} W &= Fd = (11.76 \text{ N})(0.7 \text{ m}) \\ &\approx 8.2 \text{ J} \end{aligned}$$

# Example 1(b) – *Solution*

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Here the force is given as  $F = 20$  lb, so the work done is

$$\begin{aligned} W &= Fd = (20 \text{ lb})(6 \text{ ft}) \\ &= 120 \text{ ft-lb} \end{aligned}$$

Notice that in part (b), unlike part (a), we did not have to multiply by  $g$  because we were given the *weight* (which is a force) and not the mass of the object.

# Work

Equation 2 defines work as long as the force is constant, but what happens if the force is variable? Let's suppose that the object moves along the  $x$ -axis in the positive direction, from  $x = a$  to  $x = b$ , and at each point  $x$  between  $a$  and  $b$  a force  $f(x)$  acts on the object, where  $f$  is a continuous function.

We divide the interval  $[a, b]$  into  $n$  subintervals with endpoints  $x_0, x_1, \dots, x_n$  and equal width  $\Delta x$ .

We choose a sample point  $x_i^*$  in the  $i$ th subinterval  $[x_{i-1}, x_i]$ . Then the force at that point is  $f(x_i^*)$ .

# Work

If  $n$  is large, then  $\Delta x$  is small, and since  $f$  is continuous, the values of  $f$  don't change very much over the interval  $[x_{i-1}, x_i]$ .

In other words,  $f$  is almost constant on the interval and so the work  $W_i$  that is done in moving the particle from  $x_{i-1}$  to  $x_i$  is approximately given by Equation 2:

$$W_i \approx f(x_i^*) \Delta x$$

Thus we can approximate the total work by

$$\boxed{3} \quad W \approx \sum_{i=1}^n f(x_i^*) \Delta x$$

# Work

It seems that this approximation becomes better as we make  $n$  larger. Therefore we define the **work done in moving the object from  $a$  to  $b$**  as the limit of this quantity as  $n \rightarrow \infty$ .

Since the right side of (3) is a Riemann sum, we recognize its limit as being a definite integral and so

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$$W = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x = \int_a^b f(x) dx$$

## Example 2

When a particle is located a distance  $x$  feet from the origin, a force of  $x^2 + 2x$  pounds acts on it. How much work is done in moving it from  $x = 1$  to  $x = 3$ ?

**Solution:**

$$\begin{aligned} W &= \int_1^3 (x^2 + 2x) dx = \left. \frac{x^3}{3} + x^2 \right|_1^3 \\ &= \frac{50}{3} \end{aligned}$$

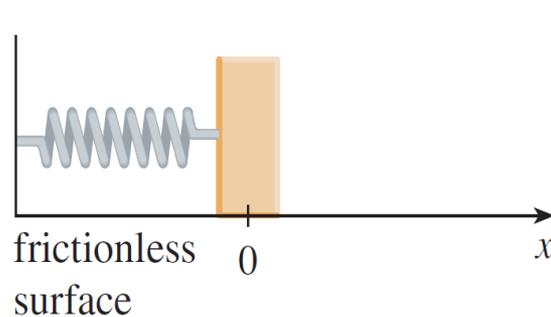
The work done is  $16\frac{2}{3}$  ft-lb.

# Work

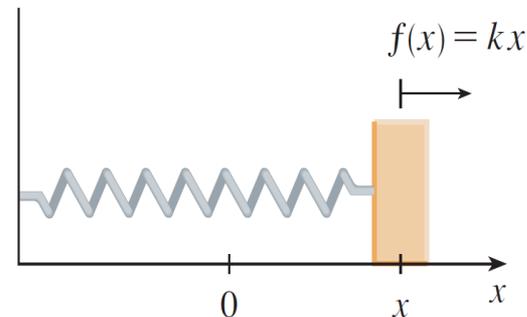
**Hooke's Law** states that the force required to maintain a spring stretched  $x$  units beyond its natural length is proportional to  $x$ :

$$f(x) = kx$$

where  $k$  is a positive constant called the **spring constant**. (see Figure 1). Hooke's Law holds provided that  $x$  is not too large.



(a) Natural position of spring



(b) Stretched position of spring

## Example 3

A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm?

### Solution:

According to Hooke's Law, the force required to hold the spring stretched  $x$  meters beyond its natural length is  $f(x) = kx$ .

## Example 3 – Solution

cont'd

When the spring is stretched from 10 cm to 15 cm, the amount stretched is 5 cm = 0.05 m. This means that  $f(0.05) = 40$ , so

$$0.05k = 40 \quad k = \frac{40}{0.05} = 800$$

Thus  $f(x) = 800x$  and the work done in stretching the spring from 15 cm to 18 cm is

$$\begin{aligned} W &= \int_{0.05}^{0.08} 800x \, dx = 800 \left. \frac{x^2}{2} \right|_{0.05}^{0.08} \\ &= 400[(0.08)^2 - 0.05^2] \\ &= 1.56 \text{ J} \end{aligned}$$

# Example 4

**EXAMPLE 4** A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?

# Example 5

**EXAMPLE 5** A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is  $1000 \text{ kg/m}^3$ .)

34. The Great Pyramid of King Khufu was built of limestone in Egypt over a 20-year time period from 2580 BC to 2560 BC. Its base is a square with side length 756 ft and its height when built was 481 ft. (It was the tallest man-made structure in the world for more than 3800 years.) The density of the limestone is about  $150 \text{ lb/ft}^3$ .
- (a) Estimate the total work done in building the pyramid.
  - (b) If each laborer worked 10 hours a day for 20 years, for 340 days a year, and did 200 ft-lb/h of work in lifting the limestone blocks into place, about how many laborers were needed to construct the pyramid?



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