Chapter 7

Techniques of Integration

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7.1 Integration by Parts

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Integration by Parts (1 of 4)

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called the rule for *integration by parts*.

The Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

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Integration by Parts (2 of 4)

In the notation for indefinite integrals this equation becomes

$$\int [f(x)g'(x) + g(x)f'(x)]\,dx = f(x)g(x)$$

or

$$\int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx = f(x)g(x)$$

We can rearrange this equation as

$$\mathbf{1} \quad \int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Formula 1 is called the **formula for integration by parts**.

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Integration by Parts (3 of 4)

It is perhaps easier to remember in the following notation.

Let $u = f(x)$ and $v = g(x)$. Then the differentials are $du = f'(x)dx$ and $dv = g'(x)dx$, so, by the Substitution Rule, the formula for integration by parts becomes

$$\mathbf{2} \quad \int u\,dv = uv - \int v\,du$$

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Example 1

Find $\int x \sin x\,dx$.

Solution Using Formula 1:

Suppose we choose $f(x) = x$ and $g'(x) = \sin x$. Then $f'(x) = 1$ and $g(x) = -\cos x$.

(For g we can choose *any* antiderivative of g'.) Thus, using Formula 1, we have

$$\begin{aligned} \int x \sin x\,dx &= f(x)g(x) - \int g(x)f'(x)\,dx \\ &= x(-\cos x) - \int (-\cos x)\,dx \\ &= -x\cos x + \int \cos x\,dx \\ &= -x\cos x + \sin x + C \end{aligned}$$

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Example 1 – *Solution* (1 of 2)

It's wise to check the answer by differentiating it. If we do so, we get $x \sin x$, as expected.

Solution Using Formula 2:

Let $u = x \quad dv = \sin x \, dx$

Then $du = dx \quad v = -\cos x$

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Example 1 – *Solution* (2 of 2)

So

$$\begin{aligned}\int x \sin x \, dx &= \int \overbrace{x}^{u} \, \overbrace{\sin x \, dx}^{dv} \\ &= \overbrace{x}^{u}\overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{u} \, \overbrace{dx}^{du} \\ &= -x\cos x + \int \cos x \, dx \\ &= -x\cos x + \sin x + C\end{aligned}$$

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Note: A mnemonic device which is helpful for selecting u when using integration by parts is the LIATE principle of precedence for u:

Logarithmic
Inverse trigonometric
Algebraic
Trigonometric
Exponential

If the integrand has several factors, then we try to choose among them a u which appears as high as possible on the list. For example, in $\int xe^{2x} \, dx$ the integrand is xe^{2x}, which is the product of an algebraic function (x) and an exponential function (e^{2x}). Since Algebraic appears before Exponential, we choose $u = x$. Sometimes the integration turns out to be similar regardless of the selection of u and dv, but it is advisable to refer to LIATE when in doubt.

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Example 2

Evaluate $\int \ln x \, dx$.

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Example 3

Find $\int t^2 e^t \, dt$.

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Example 4

Evaluate $\int e^x \sin x \, dx$.

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Example 5

Calculate $\int_0^1 \tan^{-1} x \, dx$.

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