Chapter 7
Techniques of Integration

CENGAGE

7.2 Trigonometric Integrals

CENGAGE

Trigonometric Integrals (1 of 11)

In this section we use trigonometric identities to integrate certain combinations of trigonometric functions.

We start with powers of sine and cosine.

CENGAGE

Example 2

Find $\int \sin^5 x \cos^2 x \, dx$.

Solution:

We could convert $\cos^2 x$ to $1 - \sin^2 x$, but we would be left with an expression in terms of sin *x* with no extra cos *x* factor.

Instead, we separate a single sine factor and rewrite the remaining $\sin^4 x$ factor in terms of cos *x*:

$$\begin{aligned}\sin^5 x \cos^2 x &= (\sin^2 x)^2 \cos^2 x \sin x \\ &= (1 - \cos^2 x)^2 \cos^2 x \sin x\end{aligned}$$

Example 2 – *Solution*

Substituting *u* = cos *x*, we have *du* = −sin *x dx* and so

$$\begin{aligned}\int \sin^5 x \cos^2 x \, dx &= \int (\sin^2 x)^2 \cos^2 x \sin x \, dx \\ &= \int (1 - \cos^2 x)^2 \cos^2 x \sin x \, dx \\ &= \int (1 - u^2)^2 u^2 (-du) = -\int (u^2 - 2u^4 + u^6) \, du \\ &= -\left(\frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7}\right) + C \\ &= -\tfrac{1}{3}\cos^3 x + \tfrac{2}{5}\cos^5 x - \tfrac{1}{7}\cos^7 x + C\end{aligned}$$

Example 3

Evaluate $\int_0^{\pi} \sin^2 x \, dx$.

Solution:

If we write $\sin^2 x = 1 - \cos^2 x$, the integral is no simpler to evaluate. Using the half-angle formula for $\sin^2 x$, however, we have

$$\begin{aligned}\int_0^{\pi} \sin^2 x \, dx &= \tfrac{1}{2}\int_0^{\pi} (1 - \cos 2x) \, dx \\ &= [\tfrac{1}{2}(x - \tfrac{1}{2}\sin 2x)]_0^{\pi}\end{aligned}$$

Example 3 – *Solution*

$$= \frac{1}{2}\left(\pi - \frac{1}{2}\sin 2\pi\right) - \frac{1}{2}\left(0 - \frac{1}{2}\sin 0\right)$$
$$= \frac{1}{2}\pi$$

Notice that we mentally made the substitution $u = 2x$ when integrating $\cos 2x$.

Trigonometric Integrals (2 of 11)

To summarize, we list guidelines to follow when evaluating integrals of the form $\int \sin^m x \cos^n x\, dx$, where $m \geq 0$ and $n \geq 0$ are integers.

Trigonometric Integrals (3 of 11)

Strategy for Evaluating $\int \sin^m x \cos^n x\, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x\, dx = \int \sin^m x (\cos^2 x)^k \cos x\, dx$$
$$= \int \sin^m x (1 - \sin^2 x)^k \cos x\, dx$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x\, dx = \int (\sin^2 x)^k \cos^n x \sin x\, dx$$
$$= \int (1 - \cos^2 x)^k \cos^n x \sin x\, dx$$

Trigonometric Integrals (4 of 11)

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \quad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2}\sin 2x$$

Trigonometric Integrals (5 of 11)

We can use a similar strategy to evaluate integrals of the form $\int \tan^m x \sec^n x \, dx$.

Since $\left(\frac{d}{dx}\right) \tan x = \sec^2 x$, we can separate a $\sec^2 x$ factor and convert the remaining (even) power of secant to an expression involving tangent using the identity $\sec^2 x = 1 + \tan^2 x$.

Or, since $\left(\frac{d}{dx}\right) \sec x = \sec x \tan x$, we can separate a $\sec x \tan x$ factor and convert the remaining (even) power of tangent to secant.

Example 5

Evaluate $\int \tan^6 x \sec^4 x \, dx$.

Solution:

If we separate one $\sec^2 x$ factor, we can express the remaining $\sec^2 x$ factor in terms of tangent using the identity $\sec^2 x = 1 + \tan^2 x$.

We can then evaluate the integral by substituting $u = \tan x$ so that $du = \sec^2 x dx$:

$$\int \tan^6 x \sec^4 x \, dx = \int \tan^6 x \sec^2 x \sec^2 x \, dx$$

Example 5 – *Solution*

$$= \int \tan^6 x(1+\tan^2 x)\sec^2 x\, dx$$
$$= \int u^6(1+u^2)du = \int (u^6+u^8)du$$
$$= \frac{u^7}{7}+\frac{u^9}{9}+C$$
$$= \tfrac{1}{7}\tan^7 x + \tfrac{1}{9}\tan^9 x + C$$

CENGAGE

Trigonometric Integrals (6 of 11)

The preceding examples demonstrate strategies for evaluating integrals of the form $\int \tan^m x \sec^n x\, dx$ for two cases, which we summarize here.

Strategy for Evaluating $\int \tan^m x \sec^n x\, dx$

(a) If the power of secant is even ($n = 2k$, $k \geq 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1+\tan^2 x$ to express the remaining factors in terms of tan x:

$$\int \tan^m x \sec^{2k} x\, dx = \int \tan^m x(\sec^2 x)^{k-1}\sec^2 x\, dx$$
$$= \int \tan^m x(1+\tan^2 x)^{k-1}\sec^2 x\, dx$$

Then substitute u = tan x.

CENGAGE

Trigonometric Integrals (7 of 11)

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of sec x tan x and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of sec x:

$$\int \tan^{2k+1} x \sec^n x\, dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x\, dx$$
$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x\, dx$$

Then substitute u = sec x.

CENGAGE

Trigonometric Integrals (8 of 11)

For other cases, the guidelines are not as clear-cut. We may need to use identities, integration by parts, and occasionally a little ingenuity.

We will sometimes need to be able to integrate tan *x* by using the formula given below:

$$\int \tan x \, dx = \ln|\sec x| + C$$

Trigonometric Integrals (9 of 11)

We will also need the indefinite integral of secant:

1 $$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

We could verify Formula 1 by differentiating the right side, or as follows. First we multiply numerator and denominator by sec *x* + tan *x*:

$$\int \sec x \, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$

Trigonometric Integrals (10 of 11)

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

If we substitute $u = \sec x + \tan x$, then $du = (\sec x \tan x + \sec^2 x) \, dx$, so the integral becomes

$$\int \left(\frac{1}{u}\right) du = \ln|u| + C.$$

Thus we have

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

Example 7

Find $\int \tan^3 x \, dx$.

Solution:

Here only tan x occurs, so we use $\tan^2 x = \sec^2 x - 1$ to rewrite a $\tan^2 x$ factor in terms of $\sec^2 x$:

$$\begin{aligned}\int \tan^3 x \, dx &= \int \tan x \tan^2 x \, dx \\ &= \int \tan x(\sec^2 x - 1) \, dx \\ &= \int \tan x \sec^2 x dx - \int \tan x \, dx\end{aligned}$$

CENGAGE

Example 7 – *Solution*

$$= \frac{\tan^2 x}{2} - \ln|\sec x| + C$$

In the first integral we mentally substituted $u = \tan x$ so that $du = \sec^2 x \, dx$.

CENGAGE

Trigonometric Integrals (11 of 11)

Finally, we can make use of another set of trigonometric identities:

2 To evaluate the integrals (a) $\int \sin mx \cos nx \, dx$, (b) $\int \sin mx \sin nx \, dx$, or (c) $\int \cos mx \cos nx \, dx$, use the corresponding identity:

(a) $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$

(b) $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$

(c) $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

CENGAGE

Example 9

Evaluate $\int \sin 4x \cos 5x \, dx$.

Solution:

This integral could be evaluated using integration by parts, but it's easier to use the identity in Equation 2(a) as follows:

$$\begin{aligned}\int \sin 4x \cos 5x \, dx &= \int \tfrac{1}{2}[\sin(-x) + \sin 9x] \, dx \\ &= \tfrac{1}{2}\int (-\sin x + \sin 9x) \, dx \\ &= \tfrac{1}{2}(\cos x - \tfrac{1}{9}\cos 9x) + C\end{aligned}$$