$$\int_0^{\pi/3} \boxed{\sin^3 x}\ \boxed{\cos^5 x}\ dx$$

$\sin^2 x\ \sin x$

$$\int_1^{1/2} (1-u^2)(u^5)(-du)$$

$$-\int_1^{1/2} u^5 - u^7\ du$$

$$\int_{1/2}^{1} u^5 - u^7\ du$$

$u = \cos x$

$du = -\sin x\ dx$

$$\boxed{\begin{array}{l} u(0) = \cos 0 = 1 \\ u(\pi/3) = \cos \pi/3 = 1/2 \end{array}}$$

$\sin^2 x = 1 - \cos^2 x$

u^2

$$\int_{1/2}^{1} u^5 - u^7 \, du$$

$$\left.\frac{u^6}{6} - \frac{u^8}{8}\right]_{1/2}^{1} = \left(\frac{1}{6} - \frac{1}{8}\right) - \left(\frac{1}{384} - \frac{1}{2048}\right)$$

$$= \boxed{\frac{81}{2048}}$$

In finding the area of a circle or an ellipse, an integral of the form $\int \sqrt{a^2 - x^2}\, dx$ arises, where $a > 0$. If it were $\int x\sqrt{a^2 - x^2}\, dx$, the substitution $u = a^2 - x^2$ would be effective but, as it stands, $\int \sqrt{a^2 - x^2}\, dx$ is more difficult. If we change the variable from x to θ by the substitution $x = a \sin \theta$, then the identity $1 - \sin^2\theta = \cos^2\theta$ allows us to get rid of the root sign because

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2 \cos^2\theta} = a\,|\cos \theta|$$

$$\int \sqrt{4 - x^2}\, dx$$

$$x = 2 \sin \theta$$
$$dx = 2 \cos \theta \, d\theta$$

$$\int \sqrt{4-x^2}\,dx$$

$$x = 2\sin\theta$$

$$dx = 2\cos\theta\, d\theta$$

$$\int \sqrt{4-(2\sin\theta)^2}\,(2\cos\theta)\,d\theta$$

$$\sqrt{4-4\sin^2\theta}$$

$$\sqrt{4(1-\sin^2\theta)}$$

$$\sqrt{4\cos^2\theta} = 2\cos\theta$$

$$\int (2\cos\theta)(2\cos\theta)\,d\theta$$

$$4\int \cos^2\theta \, d\theta \qquad \cos^2\theta = 1/2(1+\cos 2\theta)$$

$$2\int 1+\cos 2\theta \, d\theta$$

$$2\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

$$2\theta + \sin 2\theta + C$$

$$2\theta + \sin 2\theta + C$$

$$\boxed{2\left(\sin^{-1}\left(\frac{x}{2}\right)\right) + \sin\left(2\left(\sin^{-1}\frac{x}{2}\right)\right)}$$

$$x = 2\sin\theta$$

$$\frac{x}{2} = \sin\theta$$

$$\sin^{-1}\left(\frac{x}{2}\right) = \theta$$

$\sqrt{a^2 - x^2}$ $x = a\sin\theta$ $\cos^2\theta = 1 - \sin^2\theta$

$\sqrt{a^2 + x^2}$ $x = a\tan\theta$ $1 + \tan^2\theta = \sec^2\theta$

$\sqrt{x^2 - a^2}$ $x = a\sec\theta$ $\sec^2\theta - 1 = \tan^2\theta$

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx$$

$x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$$\int \frac{3\cos\theta}{(3\sin\theta)^2}(3\cos\theta)\,d\theta$$

$\sqrt{9-(3\sin\theta)^2}$

$\sqrt{9-9\sin^2\theta}$

$\sqrt{9(1-\sin^2\theta)}$

$\sqrt{9\cos^2\theta}$

$3\cos\theta$

$$\int \frac{3\cos\theta}{(3\sin\theta)^2}(3\cos\theta)\,d\theta = \int \frac{9\cos^2\theta}{9\sin^2\theta}\,d\theta$$

$$\int \cot^2\theta\,d\theta \qquad \cot^2\theta = \csc^2\theta - 1$$

$$\int \csc^2\theta - 1\,d\theta$$

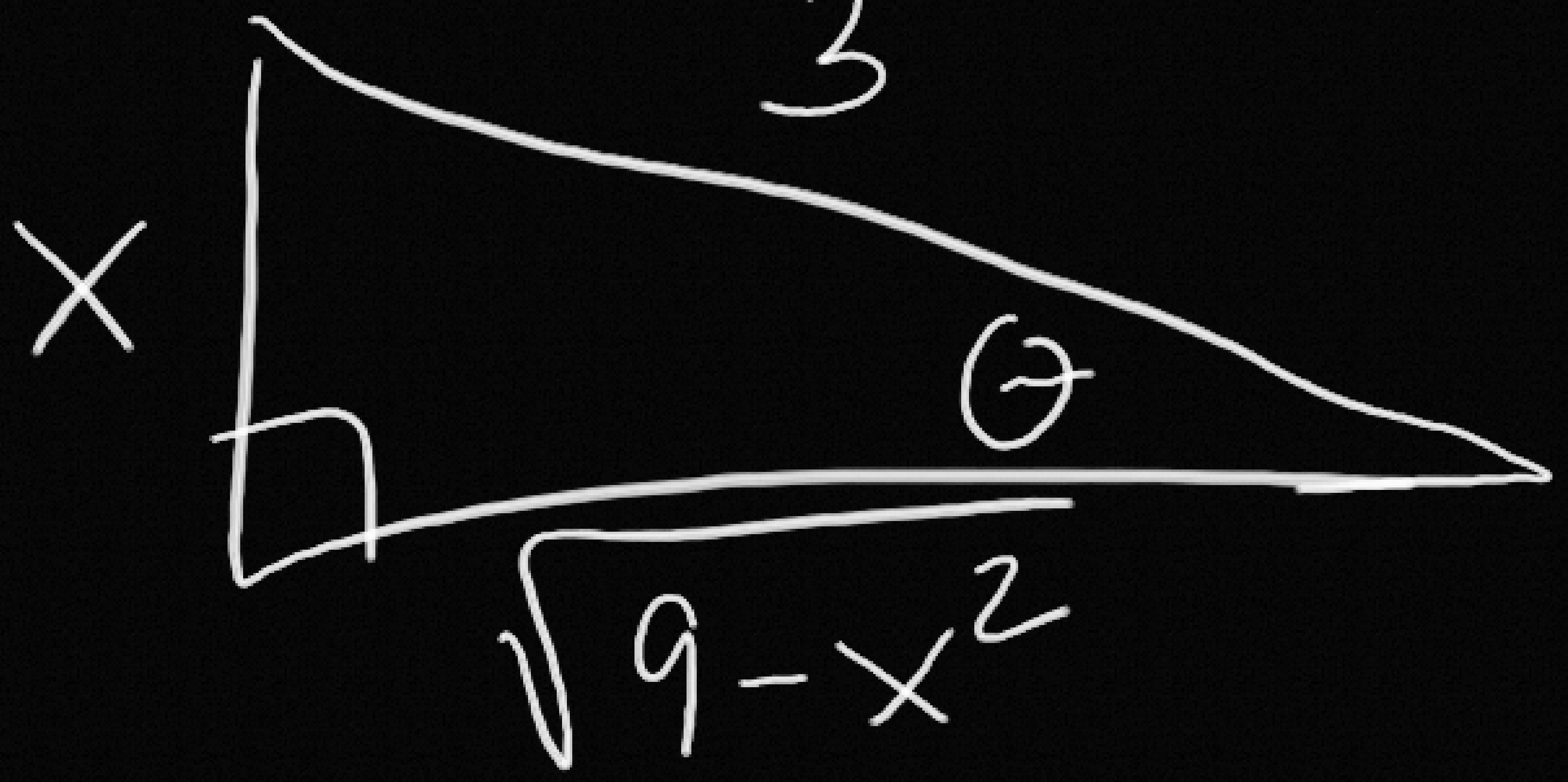
$$\cot\theta - \theta + C$$

$$\cot\theta - \theta + C$$

$$\boxed{\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\frac{x}{3} + C}$$

$$x = 3\sin\theta$$

$$\frac{x}{3} = \sin\theta$$



SOHCAHTOA

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx \qquad x = 2\tan\theta$$

$$dx = 2\sec^2\theta\,d\theta$$

$$\int \frac{2\sec^2\theta}{4\tan^2\theta\;2\sec\theta}\,d\theta \qquad \sqrt{x^2+4} = 2\sec\theta$$

$$\int \frac{\sec\theta}{4\tan^2\theta}\,d\theta = \frac{1}{4}\int \frac{1}{\cos}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}$$

$$\frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

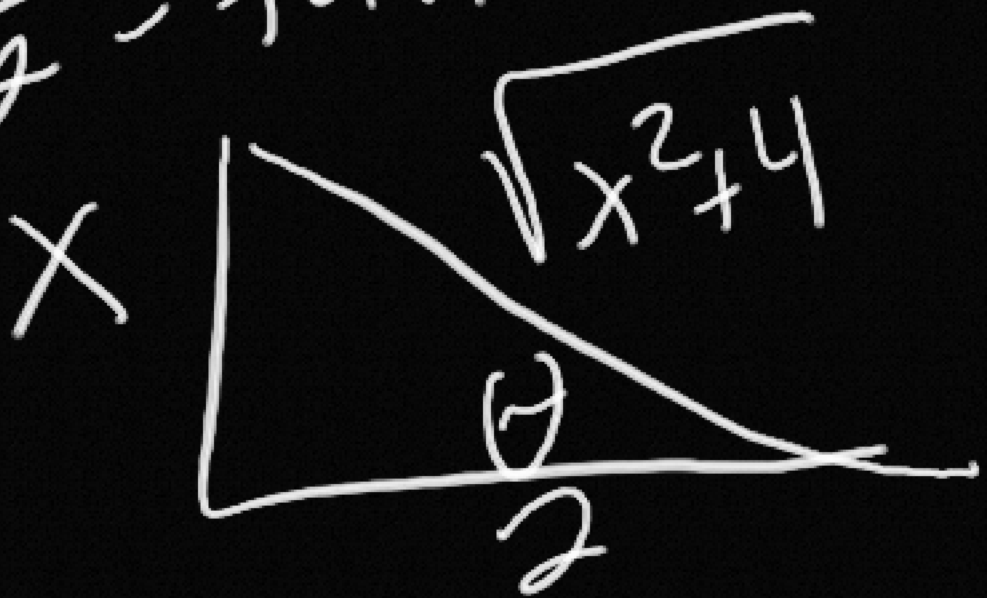
$u = \sin\theta$

$du = \cos\theta\,d\theta$

$$\frac{1}{4}\int \frac{1}{u^2}\,du = \frac{1}{4}\left(-\frac{1}{u}\right)+C = -\frac{1}{4\sin\theta}+C$$

$x = 2\tan\theta$

$\frac{x}{2} = \tan\theta$



$$\frac{-1}{4\left(\frac{x}{\sqrt{x^2+4}}\right)} = \boxed{-\frac{\sqrt{x^2+4}}{4x}+C}$$

$$\int \frac{x}{\sqrt{x^2+4}}\,dx$$

$$u = x^2 + 4$$

$$du = 2x\,dx$$

$$\frac{1}{2}\,du = x\,dx$$

$$\frac{1}{2}\int \frac{1}{u^{1/2}}\,du$$

$$\frac{1}{2}\left[\frac{1}{2}u^{1/2}\right] \rightarrow \boxed{\frac{1}{4}\sqrt{x^2+4} + C}$$

$$\int_{0}^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$$

$$0 = \frac{3}{2}\tan\theta$$

$$0 = \theta$$

$$\frac{3\sqrt{3}}{2} = \frac{3}{2}\tan\theta$$

$$\sqrt{3} = \tan\theta$$

$$\pi/3 = \theta$$

$$x = \frac{3}{2}\tan\theta$$

$$dx = 3/2\sec^2\theta\,d\theta$$

$$\sqrt{(4x^2+9)^3} = (3\sec\theta)^3$$

$$4\left(\frac{3}{2}\tan\theta\right)^2 + 9$$

$$4\left(\frac{9}{4}\tan^2\theta\right) + 9$$

$$9\tan^2\theta + 9$$

$$9(\tan^2\theta + 1)$$

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}$$

$$x = \frac{3}{2}\tan\theta$$

$$dx = \frac{3}{2}\sec^2\theta \, d\theta$$

$$\sqrt{4x^2+9} = 27\sec^3\theta$$

$$\cancel{27}/8$$

$$\int_0^{\pi/3} \frac{\left(\frac{3}{2}\tan\theta\right)^3}{\cancel{27}\sec^3\theta} \cdot \frac{3}{2}\sec^2\theta \, d\theta$$

$$x = 0 \quad \theta = 0$$

$$x = \frac{3\sqrt{3}}{2} \quad \theta = \pi/3$$

$$\int_0^{\pi/3} \frac{3 \tan^3\theta}{16\sec\theta} \, d\theta \rightarrow \frac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^3\theta} \cdot \cos\theta \, d\theta$$

$$\frac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta}\, d\theta$$

$$u = \cos\theta$$

$$du = -\sin\theta$$

$$u(0) = \cos 0 = 1$$

$$u(\pi/3) = \cos \pi/3 = 1/2$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$-\int_1^{1/2} \frac{1-u^2}{u^2}\, du$$

$$\frac{3}{16}\int_{1/2}^{1} u^{-2} - 1\, du \rightarrow \frac{3}{16}\left[-\frac{1}{u} - u\right]_{1/2}^{1} = \boxed{\frac{3}{32}}$$