$$\int\left(\frac{2}{x-1}-\frac{1}{x+2}\right)dx = \boxed{2\ln|x-1| - \ln|x+2| + C}$$

$$\int \frac{2}{x-1}\,dx - \int \frac{1}{x+2}\,dx$$

$$2\int \frac{1}{x-1}\,dx \qquad u = x-1 \qquad du = dx$$

$$2\int \frac{1}{u}\,du$$

$$2\ln|x-1| + C$$

$$\int \frac{1}{x+2} \qquad u = x+2 \qquad du = dx$$

$$\int \frac{1}{u}\,du$$

$$\ln|x+2| + C$$

$$\int \frac{x+5}{x^2+x-2}\,dx$$

$$\boxed{\frac{2}{x-1} - \frac{1}{x+2}} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

$$\frac{2(x+2)}{(x-1)(x+2)}$$

$2x+4 \quad -x+1$

$x+5$

x^2+x-2

$$\frac{x+5}{x^2+x-2} = \frac{x+5}{(x+2)(x-1)}$$

$$\frac{-1}{x+2} + \frac{2}{x-1} = x+5$$

$$A(x-1)+B(x+2)=x+5$$

$$Ax-A+Bx+2B=x+5$$

$$A+B=1$$

$$-A+2B=5$$

$$A(x-1)+B(x+2)=x+5$$

$$Ax-A+Bx+2B=x+5$$

$$\boxed{(A+B)x}+(-A+2B)=1x+5$$

$$(A+B)x=x \qquad -A+2B=5$$

$$A+B=1 \qquad A+2=1$$

$$-A+2B=5 \qquad A=-1$$

$$3B=6 \qquad B=2$$

$$\frac{3B}{3}=\frac{6}{3}$$

Find $\int \frac{x^3 + x}{x - 1}\, dx.$

$$x^2 + x + 2$$

$$x - 1 \,\big)\, x^3 + 0x^2 + x + 0$$

$$x^3 - x^2$$

$$x^2 + x + 0$$

$$-x^2 + x$$

$$2x + 0$$

$$-2x + 2$$

$$R = 2$$

$$\int x^2 + x + 2 + \frac{2}{x - 1}\, dx$$

$$\frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x - 1|$$

$$u = x - 1$$

$$du = dx$$

$$\ln|u|$$

CASE I The denominator $Q(x)$ is a product of distinct linear factors.

This means that we can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_kx + b_k)$$

where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that

$$\boxed{2} \qquad \frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx.$ $= \int \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}\,dx$

$2x^3+3x^2-2x$

$x(2x^2+3x-2)$

$x(2x-1)(x+2)$

$A(2x-1)(x+2)$

$B(x)(x+2)$

$C(x)(2x-1)$

$$x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

$$x=0 \to -1 = A(-1)(2) \to -1=-2A$$

$$\frac{1}{2} = A$$

$$x=-2 \to -1 = C(-2)(-5) \to -1 = 10C$$

$$C = -\frac{1}{10}$$

$$x = \frac{1}{2} \to \frac{1}{4} = B\left(\frac{1}{2}\right)\left(\frac{5}{2}\right)$$

$$\frac{1}{4} = \frac{5}{4}B$$

$$B = \frac{1}{5}$$

$$\int \frac{1/2}{x} + \frac{1/5}{2x-1} + \frac{-1/10}{x+2}\,dx$$

$$\boxed{\frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C}$$

$$\frac{1}{5}\int \frac{1}{2x-1}\,dx \qquad u = 2x-1$$

$$du = 2\,dx$$

$$\frac{1}{5}\left(\frac{1}{2}\right)\ln|2x-1| \qquad \frac{1}{2}du = dx$$

CASE II $Q(x)$ **is a product of linear factors, some of which are repeated.**

Suppose the first linear factor $(a_1x + b_1)$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$ in Equation 2, we would use

$$\boxed{7} \qquad \frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

By way of illustration, we could write

$$\frac{x^3 - x + 1}{x^2(x - 1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2} + \frac{E}{(x - 1)^3}$$

Find $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx.$

$$\begin{array}{r|l} & \qquad\qquad\qquad x+1 \\ \hline x^3-x^2-x+1 & x^4+0x^3-2x^2+4x+1 \\ & -x^4+x^3+x^2-x \\ \hline & \quad x^3-x^2+3x+1 \\ & \quad -x^3+x^2+x-1 \\ \hline & \qquad\qquad 4x \end{array}$$

$$\int x + 1 + \frac{4x}{x^3 - x^2 - x + 1}$$

$$x^3 - x^2 \mid -x + 1$$

$$x^2(x-1) - 1(x-1)$$

$$(x^2-1)\boxed{(x-1)}$$

$$\frac{x^2-1}{\boxed{(x-1)}(x+1)}$$

$$(x-1)^2(x+1)$$

$$\int x+1+\frac{4x}{x^3-x^2-x+1}$$

$(x-1)^2$ $(x+1)$

$$\int x+1+\boxed{\frac{4x}{(x-1)^2(x+1)}} \rightarrow \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$$

$$4x = A(x-1)(x+1)+B(x+1)+C(x-1)^2$$

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

$x=1 \Rightarrow 4 = B(2) \rightarrow B = 2$

$x=-1 \Rightarrow -4 = C(4) \rightarrow C = -1$

$x=0 \rightarrow 0 = A(-1)(1) + B(1) + C(1)$

$0 = -A + B + C$

$0 = -A + 2 - 1$

$0 = -A + 1$

$1 = A$

$$\int x+1+\frac{1}{x-1}+\frac{2}{(x-1)^2}+\frac{-1}{x+1}\,dx$$

$$\boxed{\frac{x^2}{2}+x+\ln|x-1|-\frac{2}{x-1}-\ln|x+1|+C}$$

$$2\int\frac{1}{(x-1)^2}dx \qquad u=x-1 \qquad \int\frac{1}{u^2}$$

$$du=dx$$

$$\frac{u^{-1}}{-1}=-\frac{1}{u}=\frac{-1}{x-1}$$

CASE III $Q(x)$ contains irreducible quadratic factors, none of which is repeated.
If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions in Equations 2 and 7, the expression for $R(x)/Q(x)$ will have a term of the form

9
$$\frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants to be determined. For instance, the function given by $f(x) = x/[(x - 2)(x^2 + 1)(x^2 + 4)]$ has a partial fraction decomposition of the form

$$\frac{x}{(x - 2)(x^2 + 1)(x^2 + 4)} = \frac{A}{x - 2} + \boxed{\frac{Bx + C}{x^2 + 1}} + \frac{Dx + E}{x^2 + 4}$$

The term given in (9) can be integrated by completing the square (if necessary) and using the formula

10
$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Evaluate $\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$.

$$\int \frac{2x^2 - x + 4}{x(x^2+4)} = \int \frac{A}{x} + \frac{Bx + C}{x^2+4}\,dx$$

$$\int \frac{1}{x} + \frac{x-1}{x^2+4}\,dx = \int \frac{1}{x}\,dx + \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx$$

$$= \boxed{\ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$a = 2$

$$2x^2 - x + 4 = A(x^2+4) + (Bx+C)(x)$$

$$A(x^2) + 4A + B(x^2) + C(x)$$

$$\boxed{\begin{aligned} 2 &= A + B \\ -1 &= C \\ 4 &= 4A \end{aligned}}$$

$\rightarrow B = 1$

$\rightarrow A = 1$

$$\int \frac{x}{x^2+4}\,dx \qquad u = x^2+4$$

$$du = 2x\,dx$$

$$\frac{1}{2}\int \frac{1}{u}\,du \qquad \frac{1}{2}du = x\,dx$$

$$\frac{1}{2}\ln|u| \rightarrow \frac{1}{2}\ln|x^2+4|$$

Evaluate $\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\,dx$.

$$\frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} = 1 + \frac{x - 1}{4x^2 - 4x + 3}$$

Notice that the quadratic $4x^2 - 4x + 3$ is irreducible because its discriminant is $b^2 - 4ac = -32 < 0$. This means it can't be factored, so we don't need to use the partial fraction technique.

To integrate the given function we complete the square in the denominator:

$$4x^2 - 4x + 3 = (2x - 1)^2 + 2$$

$$\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\,dx = \int \left(1 + \frac{x - 1}{4x^2 - 4x + 3}\right) dx$$

$$= x + \tfrac{1}{2}\int \frac{\frac{1}{2}(u + 1) - 1}{u^2 + 2}\,du = x + \tfrac{1}{4}\int \frac{u - 1}{u^2 + 2}\,du$$

$$= x + \tfrac{1}{4}\int \frac{u}{u^2 + 2}\,du - \tfrac{1}{4}\int \frac{1}{u^2 + 2}\,du$$

$$= x + \tfrac{1}{8}\ln(u^2 + 2) - \frac{1}{4}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$= x + \tfrac{1}{8}\ln(4x^2 - 4x + 3) - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x - 1}{\sqrt{2}}\right) + C$$

CASE IV $Q(x)$ contains a repeated irreducible quadratic factor.
If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction (9), the sum

$$\boxed{11} \qquad \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of the terms in (11) can be integrated by using a substitution or by first completing the square if necessary.

Evaluate $\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\,dx.$

SOLUTION The form of the partial fraction decomposition is

$$\frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{(x^2+1)^2}$$

Multiplying by $x(x^2+1)^2$, we have

$$\begin{aligned} -x^3 + 2x^2 - x + 1 &= A(x^2+1)^2 + (Bx + C)x(x^2+1) + (Dx + E)x \\ &= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex \\ &= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A \end{aligned}$$

If we equate coefficients, we get the system

$$A + B = 0 \qquad C = -1 \qquad 2A + B + D = 2 \qquad C + E = -1 \qquad A = 1$$

e

which has the solution $A = 1$, $B = -1$, $C = -1$, $D = 1$, and $E = 0$. Thus

$$\begin{aligned} \int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\,dx &= \int \left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}\right) dx \\ &= \int \frac{dx}{x} - \int \frac{x}{x^2+1}\,dx - \int \frac{dx}{x^2+1} + \int \frac{x\,dx}{(x^2+1)^2} \\ &= \ln|x| - \tfrac{1}{2}\ln(x^2+1) - \tan^{-1}x - \frac{1}{2(x^2+1)} + K \end{aligned}$$