Chapter 8

Further Applications of Integration

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8.2 Area of a Surface of Revolution

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Area of a Surface of Revolution (1 of 16)

A surface of revolution is formed when a curve is rotated about a line. Such a surface is the lateral boundary of a solid of revolution.

We want to define the area of a surface of revolution in such a way that it corresponds to our intuition.

If the surface area is *A*, we can imagine that painting the surface would require the same amount of paint as does a flat region with area *A*.

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Area of a Surface of Revolution (2 of 16)

Let's start with some simple surfaces. The lateral surface area of a circular cylinder with radius *r* and height *h* is taken to be $A = 2\pi rh$ because we can imagine cutting the cylinder and unrolling it (as in Figure 1) to obtain a rectangle with dimensions $2\pi r$ and *h*.

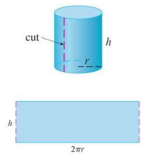


Figure 1

Area of a Surface of Revolution (3 of 16)

Likewise, we can take a circular cone with base radius *r* and slant height *l*, cut it along the dashed line in Figure 2, and flatten it to form a sector of a circle with radius *l* and central angle $\theta = \frac{2\pi r}{l}$.

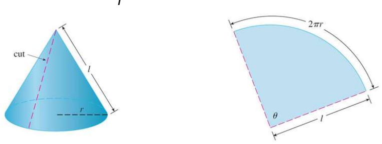


Figure 2

Area of a Surface of Revolution (4 of 16)

We know that, in general, the area of a sector of a circle with radius *l* and angle *θ* is $\frac{1}{2}l^2\theta$ and so in this case the area is

$$A = \frac{1}{2}l^2\theta = \frac{1}{2}l^2\left(\frac{2\pi r}{l}\right) = \pi rl$$

Therefore we define the lateral surface area of a cone to be $A = \pi rl$.

What about more complicated surfaces of revolution? If we follow the strategy we used with arc length, we can approximate the original curve by a polygon.

Area of a Surface of Revolution (5 of 16)

When this polygon is rotated about an axis, it creates a simpler surface whose surface area approximates the actual surface area.

By taking a limit, we can determine the exact surface area.

The approximating surface, then, consists of a number of *bands*, each formed by rotating a line segment about an axis.

Area of a Surface of Revolution (6 of 16)

To find the surface area, each of these bands can be considered a portion of a circular cone, as shown in Figure 3.

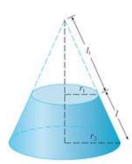


Figure 3

Area of a Surface of Revolution (7 of 16)

The area of the band (or frustum of a cone) with slant height l and upper and lower radii r_1 and r_2 is found by subtracting the areas of two cones:

$$1 \quad A = \pi r_2(l_1 + l) - \pi r_1 l_1 = \pi\left[(r_2 - r_1)l_1 + r_2 l\right]$$

From similar triangles we have

$$\frac{l_1}{r_1} = \frac{l_1 + l}{r_2}$$

which gives

$$r_2 l_1 = r_1 l_1 + r_1 l \quad \text{or} \quad (r_2 - r_1)l_1 = r_1 l$$

Area of a Surface of Revolution (8 of 16)

Putting this in Equation 1, we get

$$A = \pi(r_1 l + r_2 l)$$

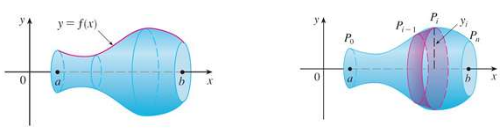
or

$$2 \quad A = 2\pi r l$$

where $r = \frac{1}{2}(r_1 + r_2)$ is the average radius of the band.

Area of a Surface of Revolution (9 of 16)

Now we apply this formula to our strategy. Consider the surface shown in Figure 4, which is obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis, where f is positive and has a continuous derivative.



(a) Surface of revolution (b) Approximating band

Figure 4

Area of a Surface of Revolution (10 of 16)

In order to define its surface area, we divide the interval [a, b] into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx, as we did in determining arc length.

If $y_i = f(x_i)$, then the point $P_i(x_i, y_i)$ lies on the curve.

The part of the surface between x_{i-1} and x_i is approximated by taking the line segment $P_{i-1}P_i$ and rotating it about the x-axis.

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The result is a band with slant height $l = |P_{i-1}P_i|$ and average radius $r = \frac{1}{2}(y_{i-1} + y_i)$ so, by Formula 2, its surface area is

$$2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i|$$

As in the proof, We have

$$|P_{i-1}P_i| = \sqrt{1 + [f'(x_i^*)]^2} \, \Delta x$$

where x_i^* is some number in $[x_{i-1}, x_i]$.

Area of a Surface of Revolution (12 of 16)

When Δx is small, we have $y_i = f(x_i) \approx f(x_i^*)$ and also $y_{i-1} = f(x_{i-1}) \approx f(x_i^*)$, since f is continuous.
Therefore

$$2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i| \approx 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2} \, \Delta x$$

and so an approximation to what we think of as the area of the complete surface of revolution is

$$3 \quad \sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2} \, \Delta x$$

Area of a Surface of Revolution (13 of 16)

This approximation appears to become better as $n \to \infty$ and, recognizing (3) as a Riemann sum for the function $g(x) = 2\pi f(x)\sqrt{1 + [f'(x)]^2}$, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2} \, \Delta x = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx$$

Therefore, in the case where f is positive and has a continuous derivative, we define the **surface area** of the surface obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis as

$$4 \quad S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx$$

Area of a Surface of Revolution (14 of 16)

With the Leibniz notation for derivatives, this formula becomes

$$5 \quad S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If the curve is described as $x = g(y)$, $c \le y \le d$, then the formula for surface area becomes

$$6 \quad S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Area of a Surface of Revolution (15 of 16)

Now both Formulas 5 and 6 can be summarized symbolically, using the notation for arc length, as

$$7 \quad S = \int 2\pi y \; ds$$

For rotation about the y-axis, the surface area formula becomes

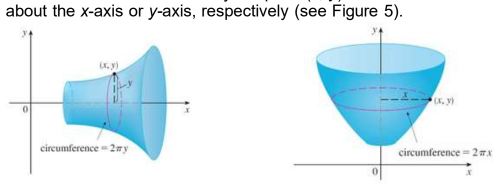
$$8 \quad S = \int 2\pi x \; ds$$

where, as before, we can use either

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{or} \quad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Area of a Surface of Revolution (16 of 16)

These formulas can be remembered by thinking of $2\pi y$ or $2\pi x$ as the circumference of a circle traced out by the point (x, y) on the curve as it is rotated about the x-axis or y-axis, respectively (see Figure 5).



(a) Rotation about x-axis: $S = \int 2\pi y\, ds$ (b) Rotation about y-axis: $S = \int 2\pi x\, ds$

Figure 5

Example 1

The curve $y = \sqrt{4 - x^2}$, $-1 \le x \le 1$, is an arc of the circle $x^2 + y^2 = 4$.

Find the area of the surface obtained by rotating this arc about the *x*-axis. (The surface is a portion of a sphere of radius 2. See Figure 6.)

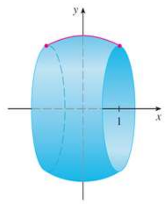


Figure 6

Example 1 – *Solution* (1 of 2)

We have

$$\frac{dy}{dx} = \frac{1}{2}\left(4 - x^2\right)^{-1/2}(-2x) = \frac{-x}{\sqrt{4 - x^2}}$$

and so, by Formula 5, the surface area is

$$S = \int_{-1}^{1} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_{-1}^{1} \sqrt{4 - x^2} \sqrt{1 + \frac{x^2}{4 - x^2}} dx$$

Example 1 – *Solution* (2 of 2)

$$= 2\pi \int_{-1}^{1} \sqrt{4 - x^2} \sqrt{\frac{4 - x^2 + x^2}{4 - x^2}} dx$$

$$= 2\pi \int_{-1}^{1} \sqrt{4 - x^2} \frac{2}{\sqrt{4 - x^2}} dx$$

$$= 4\pi \int_{-1}^{1} 1\, dx$$

$$= 4\pi(2)$$

$$= 8\pi$$

Example 3

Find the area of the surface generated by rotating the curve $y = e^x$, $0 \le x \le 1$, about the *x*-axis.

Solution:

Using Formula 5 with

$$y = e^x \quad \text{and} \quad \frac{dy}{dx} = e^x$$

we have

$$S = \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$= 2\pi \int_0^1 e^x \sqrt{1 + e^{2x}}\, dx$$

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Example 3 – *Solution* (1 of 2)

$$= 2\pi \int_1^e \sqrt{1 + u^2}\, du \quad (\text{where } u = e^x)$$
$$= 2\pi \int_{\frac{\pi}{4}}^{\alpha} \sec^3 \theta\, d\theta \quad (\text{where } u = \tan\theta \text{ and } \alpha = \tan^{-1} e)$$
$$= 2\pi \cdot \frac{1}{2}\left[\sec\theta \tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right]_{\frac{\pi}{4}}^{\alpha}$$
$$= \pi\left[\sec\alpha \tan\alpha + \ln(\sec\alpha + \tan\alpha) - \sqrt{2} - \ln\left(\sqrt{2} + 1\right)\right]$$

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Example 3 – *Solution* (2 of 2)

Since tanα = *e*, we have $\sec^2\alpha = 1 + \tan^2\alpha = 1 + e^2$ and

$$S = \pi\left[e\sqrt{1 + e^2} + \ln\left(e + \sqrt{1 + e^2}\right) - \sqrt{2} - \ln\left(\sqrt{2} + 1\right)\right]$$

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