Chapter 8

Further Applications of Integration

8.3 Applications to Physics and Engineering

Applications to Physics and Engineering

Among the many applications of integral calculus to physics and engineering, we consider two here: force due to water pressure and centers of mass.

Hydrostatic Pressure and Force

Hydrostatic Pressure and Force (1 of 5)

Deep-sea divers realize that water pressure increases as they dive deeper. This is because the weight of the water above them increases.

In general, suppose that a thin horizontal plate with area A square meters is submerged in a fluid of density ρ kilograms per cubic meter at a depth d meters below the surface of the fluid as in Figure 1.

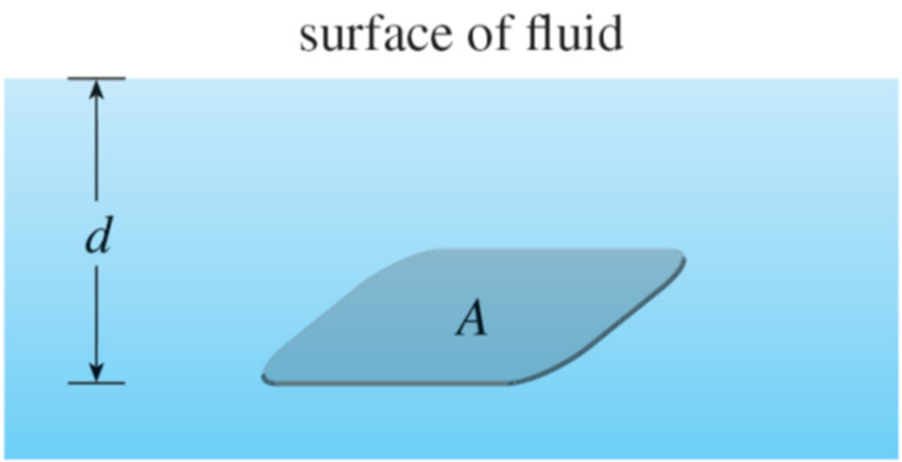

Figure 1

Hydrostatic Pressure and Force (2 of 5)

The fluid directly above the plate has volume $V = Ad$, so its mass is $m = \rho V = \rho Ad$. The force exerted by the fluid on the plate is therefore

$$F = mg = \rho g A d$$

where g is the acceleration due to gravity. The **pressure** P on the plate is defined to be the force per unit area:

$$P = \frac{F}{A} = \rho g d$$

The SI unit for measuring pressure is a newton per square meter, which is called a pascal

$\left(\text{abbrevation:} 1\,\text{N}/\text{m}^2 = 1\,\text{Pa}\right)$

Hydrostatic Pressure and Force (3 of 5)

Since this is a small unit, the kilopascal (kPa) is often used.

For instance, because the density of water is $\rho = 1000 \text{ kg/m}^3$, the pressure at the bottom of a swimming pool 2 m deep is

$$\begin{aligned} P = \rho g d &= 1000 \text{ kg/m}^3 \times 9.8 \text{ m/s}^2 \times 2\text{m} \\ &= 19{,}600 \text{ pa} \\ &= 19.6 \text{ kPa} \end{aligned}$$

Hydrostatic Pressure and Force (4 of 5)

An important principle of fluid pressure is the experimentally verified fact that *at any point in a liquid the pressure is the same in all directions.*

(A diver feels the same pressure on nose and both ears.)

Thus the pressure in *any* direction at a depth d in a fluid with mass density ρ is given by

$$\mathbf{1} \quad P = \rho g d = \delta d$$

Hydrostatic Pressure and Force (5 of 5)

This helps us determine the hydrostatic force against a *vertical* plate or wall or dam in a fluid.

This is not a straightforward problem because the pressure is not constant but increases as the depth increases

Example 1

A dam has the shape of the trapezoid shown in Figure 2. The height is 20 m and the width is 50 m at the top and 30 m at the bottom. Find the force on the dam due to hydrostatic pressure if the water level is 4 m from the top of the dam.

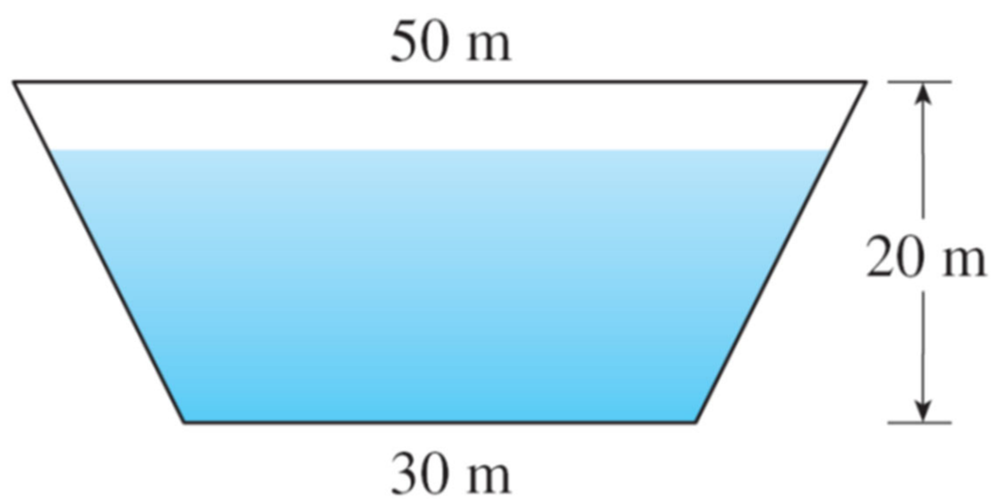

Figure 2

Example 1 – *Solution* (1 of 6)

We choose a vertical x-axis with origin at the surface of the water and directed downward as in Figure 3(a).

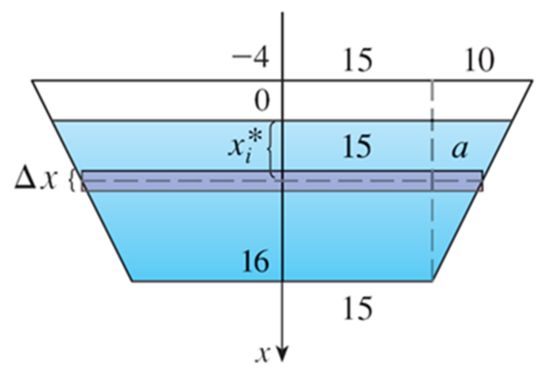

Figure 3(a)

The depth of the water is 16 m, so we divide the interval [0, 16] into subintervals of equal length with endpoints x_i and we choose $x_i^* \in [x_{i-1}, x_i]$.

Example 1 – *Solution* (2 of 6)

The *i*th horizontal strip of the dam is approximated by a rectangle with height Δx and width w_i, where, from similar triangles in Figure 3(b),

$$\frac{a}{16 - x_i^*} = \frac{10}{20}$$

or

$$a = \frac{16 - x_i^*}{2} = 8 - \frac{x_i^*}{2}$$

and so

$$W_i = 2(15 + a)$$

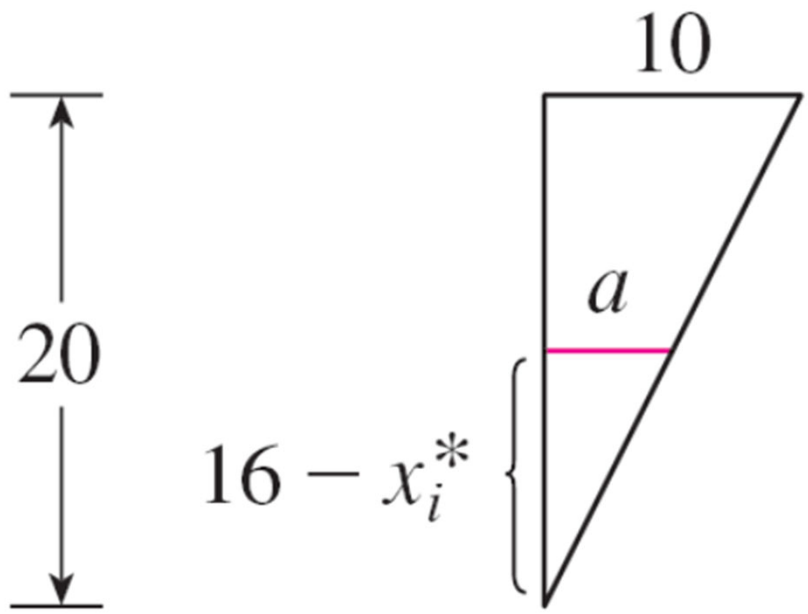

Figure 3(b)

Example 1 – *Solution* (3 of 6)

$$= 2\left(15 + 8 - \tfrac{1}{2}\, x_i^*\right)$$
$$= 46 - x_i^*$$

If A_i is the area of the *i*th strip, then

$$\begin{aligned} A_i &\approx w_i \Delta x \\ &= \left(46 - x_i^*\right)\Delta x \end{aligned}$$

Example 1 – *Solution* (4 of 6)

If Δx is small, then the pressure P_i on the ith strip is almost constant and we can use Equation 1 to write

$$P_i \approx 1000 g x_i^*$$

The hydrostatic force F_i acting on the ith strip is the product of the pressure and the area:

$$\begin{aligned} F_i &= P_i A_i \\ &\approx 1000 g x_i^* \left(46 - x_i^*\right) \Delta x \end{aligned}$$

Example 1 – *Solution* (5 of 6)

Adding these forces and taking the limit as $n \to \infty$, we obtain the total hydrostatic force on the dam:

$$
\begin{aligned}
F &= \lim_{n \to \infty} \sum_{i=1}^{n} 1000 g x_i^* \left(46 - x_i^*\right) \Delta x \\
&= \int_0^{16} 1000 g x \left(46 - x\right) dx \\
&= 1000(9.8) \int_0^{16} \left(46x - x^2\right) dx
\end{aligned}
$$

Example 1 – *Solution* (6 of 6)

$$= 9800\left[23x^2 - \frac{x^3}{3}\right]_0^{16}$$

$$\approx 4.43 \times 10^7\,\text{N}$$

Moments and Centers of Mass

Moments and Centers of Mass (1 of 9)

Our main objective here is to find the point *P* on which a thin plate of any given shape balances horizontally as in Figure 5.

This point is called the **center of mass** (or center of gravity) of the plate.

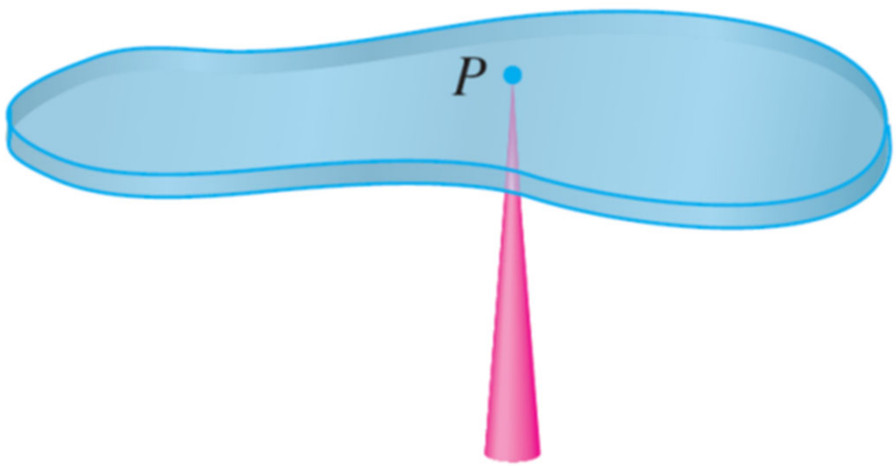

Figure 5

Moments and Centers of Mass (2 of 9)

We first consider the simpler situation illustrated in Figure 6, where two masses m_1 and m_2 are attached to a rod of negligible mass on opposite sides of a fulcrum and at distances d_1 and d_2 from the fulcrum.

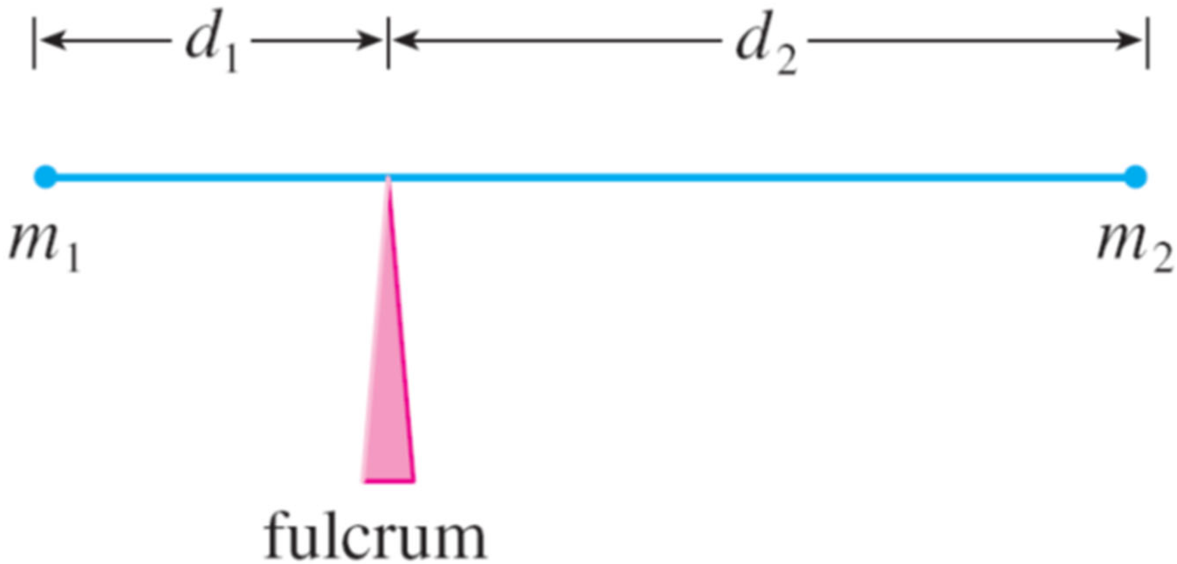

Figure 6

Moments and Centers of Mass (3 of 9)

The rod will balance if

$$\mathbf{2} \quad m_1 d_1 = m_2 d_2$$

This is an experimental fact discovered by Archimedes and called the Law of the Lever. (Think of a lighter person balancing a heavier one on a seesaw by sitting farther away from the center.)

Now suppose that the rod lies along the x-axis with m_1 at x_1 and m_2 at x_2 and the center of mass at $\bar{x}$.

Moments and Centers of Mass (4 of 9)

If we compare Figures 6 and 7, we see that $d_1 = \bar{x} - x_1$ and $d_2 = x_2 - \bar{x}$ and so Equation 2 gives

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$m_1\bar{x} + m_2\bar{x} = m_1x_1 + m_2x_2$$

3 $$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

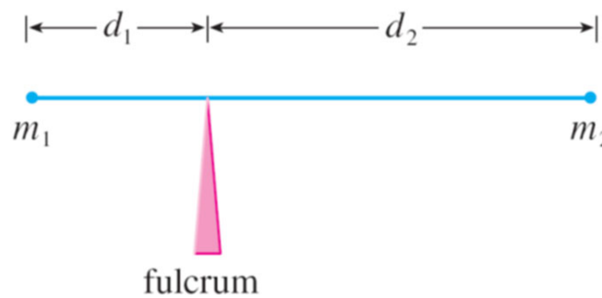

Figure 6

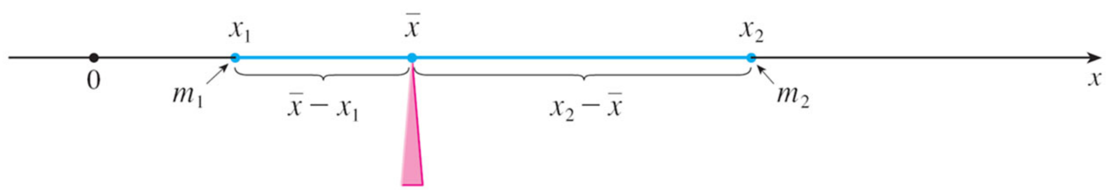

Figure 7

Moments and Centers of Mass (5 of 9)

The numbers m_1x_1 and m_2x_2 are called the **moments** of the masses m_1 and m_2 (with respect to the origin), and Equation 3 says that the center of mass $\bar{x}$ is obtained by adding the moments of the masses and dividing by the total mass $m = m_1 + m_2$

Moments and Centers of Mass (6 of 9)

In general, if we have a system of n particles with masses $m_1, m_2, \ldots, m_n$ located at the points $x_1, x_2, \ldots, x_n$ on the x-axis, it can be shown similarly that the center of mass of the system is located at

$$\mathbf{4} \quad \bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i x_i}{m}$$

where $m = \sum m_i$ is the total mass of the system, and the sum of the individual moments

$$m = \sum_{i=1}^{n} m_i x_i$$

is called the **moment of the system about the origin.**

Moments and Centers of Mass (7 of 9)

Then Equation 4 could be rewritten as $m\bar{x} = M$, which says that if the total mass were considered as being concentrated at the center of mass $\bar{x}$, then its moment would be the same as the moment of the system.

Now we consider a system of n particles with masses $m_1, m_2, \ldots, m_n$ located at the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the xy-plane as shown in Figure 8.

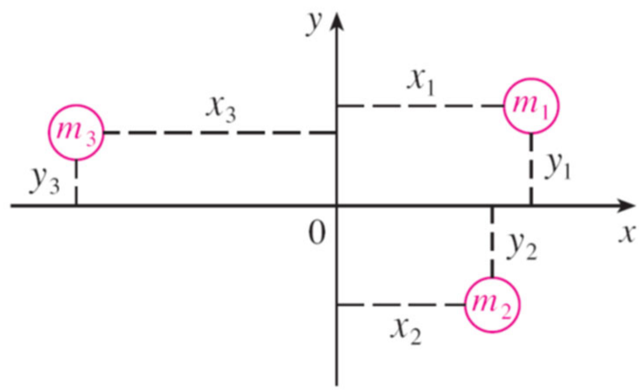

Figure 8

Moments and Centers of Mass (8 of 9)

By analogy with the one-dimensional case, we define the **moment of the system about the *y*-axis** to be

$$\mathbf{5} \quad M_y = \sum_{i=1}^{n} m_i x_i$$

and the **moment of the system about the *x*-axis** as

$$\mathbf{6} \quad M_x = \sum_{i=1}^{n} m_i y_i$$

Then M_y measures the tendency of the system to rotate about the *y*-axis and M_x measures the tendency to rotate about the *x*-axis.

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Moments and Centers of Mass (9 of 9)

As in the one-dimensional case, the coordinates $(\bar{x}, \bar{y})$ of the center of mass are given in terms of the moments by the formulas

$$\textbf{7} \quad \bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}$$

where $m = \Sigma m_i$ is the total mass. Since $m\bar{x} = M_y$ and $m\bar{y} = M_y$, the center of mass $(\bar{x}, \bar{y})$ is the point where a single particle of mass m would have the the same moments as the system.

Example 3

Find the moments and center of mass of the system of objects that have masses 3, 4, and 8 at the points (−1, 1), (2, −1), and (3, 2), respectively.

Solution:

We use Equations 5 and 6 to compute the moments:

$$M_Y = 3(-1) + 4(2) + 8(3) = 29$$
$$M_x = 3(1) + 4(-1) + 8(2) = 15$$

Example 3 – *Solution* (1 of 2)

Since $m = 3 + 4 + 8 = 15$, we use Equations 7 to obtain

$$\bar{x} = \frac{M_y}{m} = \frac{29}{15}$$

$$\bar{y} = \frac{M_x}{m} = \frac{15}{15} = 1$$

Example 3 – *Solution* (2 of 2)

Thus the center of mass is $\left(1\frac{14}{15}, 1\right)$. (See Figure 9.)

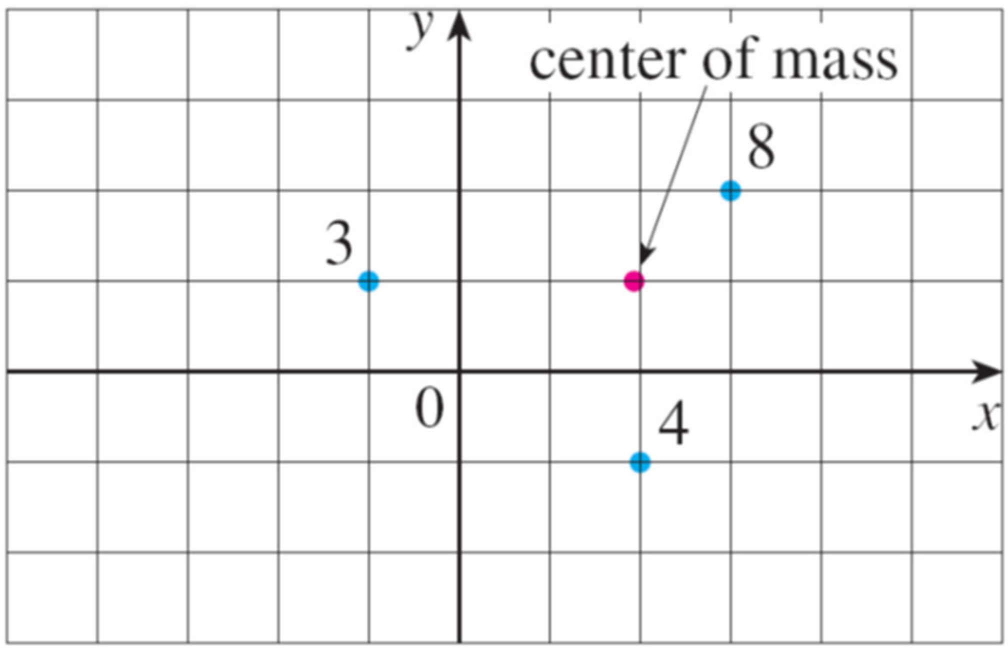

Figure 9

Moments and Centers of Mass (1 of 10)

Next we consider a flat plate (called a *lamina*) with uniform density ρ that occupies a region $\mathfrak{R}$ of the plane.

We wish to locate the center of mass of the plate, which is called the **centroid** of $\mathfrak{R}$.

In doing so we use the following physical principles: The **symmetry principle** says that if $\mathfrak{R}$ is symmetric about a line *l*, then the centroid of $\mathfrak{R}$ lies on *l*. (If $\mathfrak{R}$ is reflected about *l*, then $\mathfrak{R}$ remains the same so its centroid remains fixed. But the only fixed points lie on *l*.)

Thus the centroid of a rectangle is its center.

Moments and Centers of Mass (2 of 10)

Moments should be defined so that if the entire mass of a region is concentrated at the center of mass, then its moments remain unchanged.

Also, the moment of the union of two nonoverlapping regions should be the sum of the moments of the individual regions.

Suppose that the region $\mathcal{R}$ is of the type shown in Figure 10(a); that is, $\mathcal{R}$ lies between the lines $x = a$ and $x = b$, above the x-axis, and beneath the graph of f, where f is a continuous function.

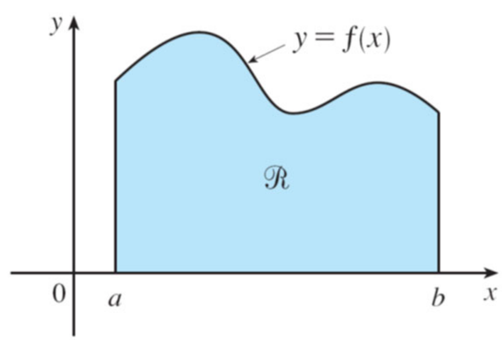

Figure 10(a)

Moments and Centers of Mass (3 of 10)

We divide the interval [a, b] into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx. We choose the sample point x_i^* to be the midpoint $\bar{x}_i$ of the ith subinterval, that is, $\bar{x}_i = \frac{(x_{i-1} + x_i)}{2}$.

This determines the polygonal approximation to $\mathfrak{R}$ shown in Figure 10(b).

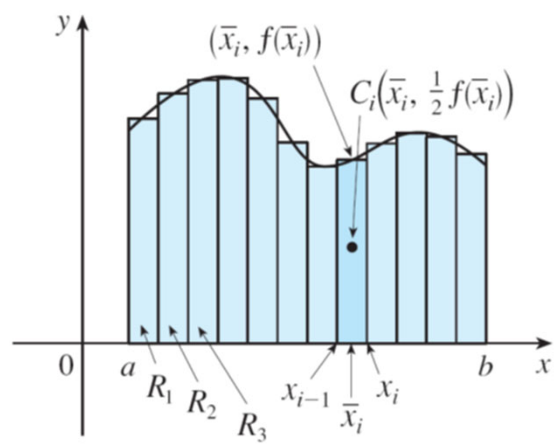

Figure 10(b)

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Moments and Centers of Mass (4 of 10)

The centroid of the *i*th approximating rectangle R_i is its center $C_i\left(\bar{x}_i,\ \frac{1}{2}f(\bar{x}_i)\right)$. Its area is $f(\bar{x}_i)\Delta x$, so its mass is

$$\rho f(\bar{x}_i)\Delta x$$

The moment of R_i about the *y*-axis is the product of its mass and the distance from C_i to the *y*-axis, which is $\bar{x}_i$. Thus

$$M_y(R_i) = \left[\rho f(\bar{x}_i)\Delta x\right]\bar{x}_i = \rho \bar{x}_i f(\bar{x}_i)\Delta x$$

Moments and Centers of Mass (5 of 10)

Adding these moments, we obtain the moment of the polygonal approximation to $\mathfrak{R}$, and then by taking the limit as $n \to \infty$ we obtain the moment of $\mathfrak{R}$ itself about the y-axis:

$$M_y = \lim_{n \to \infty} \sum_{i=1}^{n} \rho \overline{x}_i f\left(\overline{x}_i\right) \Delta x = \rho \int_a^b x\, f\left(x\right) dx$$

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Moments and Centers of Mass (6 of 10)

In a similar fashion we compute the moment of R_i about the x-axis as the product of its mass and the distance from C_i to the x-axis:

$$M_x(R_i) = \left[\rho f(\bar{x}_i)\Delta x\right]\frac{1}{2}f(\bar{x}_i) = \rho \cdot \frac{1}{2}\left[f(\bar{x}_i)\right]^2 \Delta x$$

Again we add these moments and take the limit to obtain the moment of $\mathfrak{R}$ about the x-axis:

$$M_x = \lim_{n\to\infty}\sum_{i=1}^{n}\rho \cdot \frac{1}{2}\left[f(\bar{x}_i)\right]^2 \Delta x = \rho\int_a^b \frac{1}{2}\left[f(x)\right]^2 dx$$

Moments and Centers of Mass (7 of 10)

Just as for systems of particles, the center of mass of the plate is defined so that $m\overline{x} = m_y$ and $m\overline{y} = m_x$. But the mass of the plate is the product of its density and its area:

$$m = \rho A = \rho \int_a^b f(x)\,dx$$

and so

$$\overline{x} = \frac{M_y}{m} = \frac{\rho \int_a^b x f(x)\,dx}{\rho \int_a^b f(x)\,dx} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx}$$

$$\overline{y} = \frac{M_x}{m} = \frac{\rho \int_a^b \frac{1}{2}\left[f(x)\right]^2 dx}{\rho \int_a^b f(x)\,dx} = \frac{\int_a^b \frac{1}{2}\left[f(x)\right]^2 dx}{\int_a^b f(x)\,dx}$$

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Moments and Centers of Mass (8 of 10)

Notice the cancellation of the ρ's. The location of the center of mass is independent of the density.

In summary, the center of mass of the plate (or the centroid of $\mathcal{R}$) is located at the point $(\bar{x}, \bar{y})$, where

8 $$\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[f(x)\right]^2 dx$$

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Moments and Centers of Mass

If the region $\mathscr{R}$ lies between two curves $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$, as illustrated in Figure 13, then the same sort of argument that led to Formulas 8 can be used to show that the centroid of $\mathscr{R}$ is $(\bar{x}, \bar{y})$, where

9 $$\bar{x} = \frac{1}{A}\int_a^b x\left[f(x) - g(x)\right]dx$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left\{\left[f(x)\right]^2 - \left[g(x)\right]^2\right\}dx$$

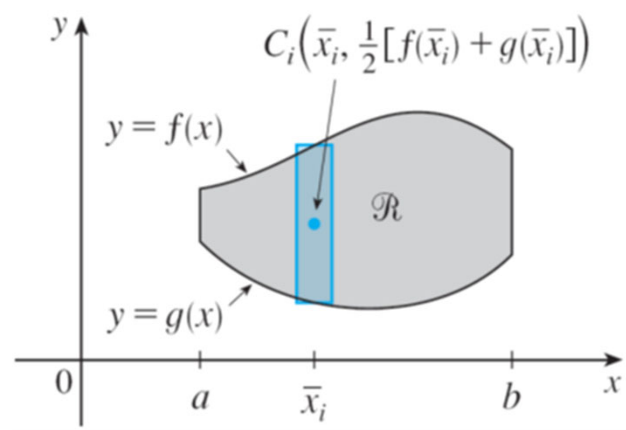

Figure 13

Moments and Centers of Mass (10 of 10)

We end this section by showing a surprising connection between centroids and volumes of revolution.

Theorem of Pappus Let $\mathcal{R}$ be a plane region that lies entirely on one side of a line *l* in the plane. If $\mathcal{R}$ is rotated about *l*, then the volume of the resulting solid is the product of the area *A* of $\mathcal{R}$ and the distance *d* traveled by the centroid of $\mathcal{R}$.

Example 7

A torus is formed by rotating a circle of radius *r* about a line in the plane of the circle that is a distance *R* (> *r*) from the center of the circle. Find the volume of the torus.

Solution:

The circle has area $A = \pi r^2$. By the symmetry principle, its centroid is its center and so the distance traveled by the centroid during a rotation is $d = 2\pi R$. Therefore, by the Theorem of Pappus, the volume of the torus is

$$V = Ad = (2\pi R)(\pi r^2) = 2\pi^2 r^2 R$$