Probability and Calculus

$$P(a \leqslant X \leqslant b) = \int_a^b f(x)\,dx$$

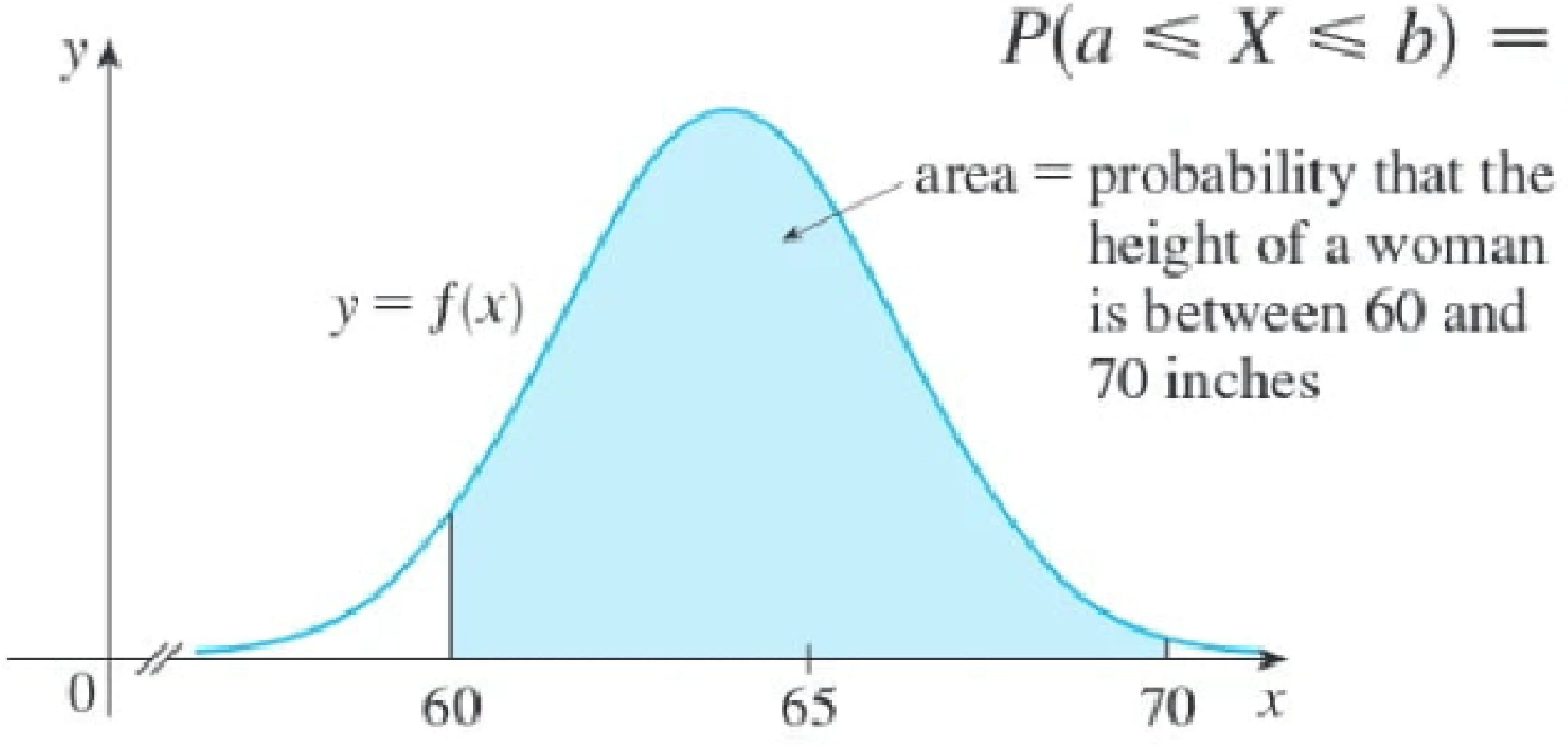


FIGURE 1
Probability density function for the height of an adult female

In general, the probability density function f of a random variable X satisfies the condition $f(x) \geqslant 0$ for all x. Because probabilities are measured on a scale from 0 to 1, it follows that

2
$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

EXAMPLE 1 Let $f(x) = 0.006x(10 - x)$ for $0 \leqslant x \leqslant 10$ and $f(x) = 0$ for all other values of x.

(a) Verify that f is a probability density function.

(b) Find $P(4 \leqslant X \leqslant 8)$.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^{10} 0.006x(10-x)\,dx$$

verified

$$0.006 \int_0^{10} 10x - x^2\,dx$$

$$0.006\left[\frac{10x^2}{2} - \frac{x^3}{3}\right]_0^{10} = 0.006\left(500 - \frac{1000}{3}\right)$$

$$0.006\left[5x^2 - \frac{x^3}{3}\right]_4^8 = 0.544$$

54.4%

Let $f(x) = 30x^2(1-x)^2$ for $0 \leqslant x \leqslant 1$ and $f(x) = 0$ for all other values of x.

(a) Verify that f is a probability density function.

(b) Find $P\left(X \leqslant \frac{1}{3}\right)$.

$1 - 2x + x^2$

$$\int_0^1 30x^2(1-x)^2\,dx$$

$$30\int_0^1 x^2 - 2x^3 + x^4\,dx$$

$$30\left[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right]_0^1$$

$$10x^3 - 15x^4 + 6x^5\Big]_0^1 = 1$$

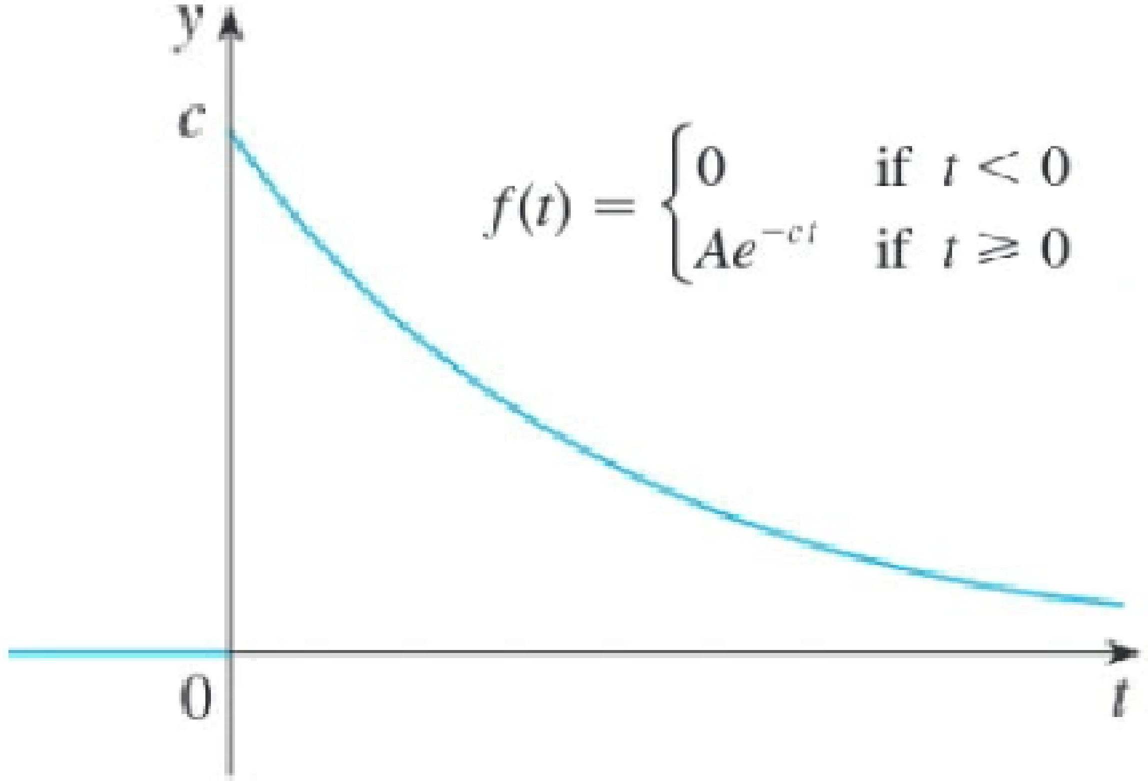
verified

$$P(x \leq 1/3)$$

$$\int_0^{1/3} f(x)\,dx = \frac{17}{81}$$

$$= 0.210$$

EXAMPLE 2 Phenomena such as waiting times and equipment failure times are commonly modeled by exponentially decreasing probability density functions. Find the exact form of such a function.



$$\int_0^\infty A e^{-ct}\,dt$$

$$1 = \int_0^\infty A e^{-ct}\,dt$$

$$\lim_{x \to \infty} \int_0^x A e^{-ct}\,dt$$

$$\lim_{x \to \infty} \left[-\frac{A}{c} e^{-ct} \right]_0^x = \left(\lim_{x \to \infty} -\frac{A}{c} e^{-cx} \right) + \frac{A}{c}$$

$$\int e^{-ct}\,dt$$

$$\boxed{\begin{aligned} u &= -ct \\ du &= -c\,dt \\ -\frac{1}{c}\,du &= dt \end{aligned}}$$

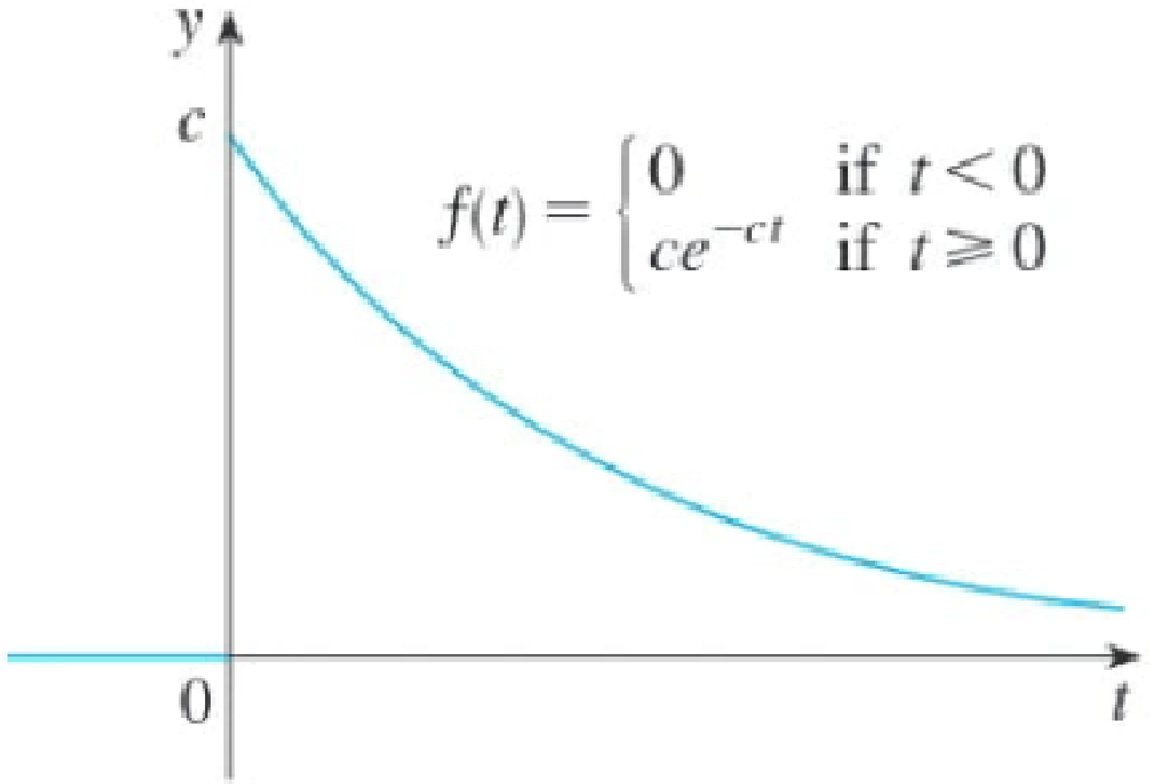
$$-\frac{1}{c}\int e^u\,du$$

$$-\frac{1}{c} e^u$$

$$-\frac{1}{c} e^{-ct}$$

EXAMPLE 2 Phenomena such as waiting times and equipment failure times are commonly modeled by exponentially decreasing probability density functions. Find the exact form of such a function.

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ Ae^{-ct} & \text{if } t \geq 0 \end{cases}$$



$$\lim_{x\to\infty}\left[\frac{-A}{c}e^{-ct}\right]_0^x$$

$$\boxed{\left(\lim_{x\to\infty}\frac{-A}{c}e^{-cx}\right)}+\frac{A}{c}$$

$$0+\frac{A}{c}=1$$

$$\frac{A}{c}=1 \Rightarrow \boxed{A=c}$$

Average Values

In general, the **mean** of any probability density function f is defined to be

$$\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

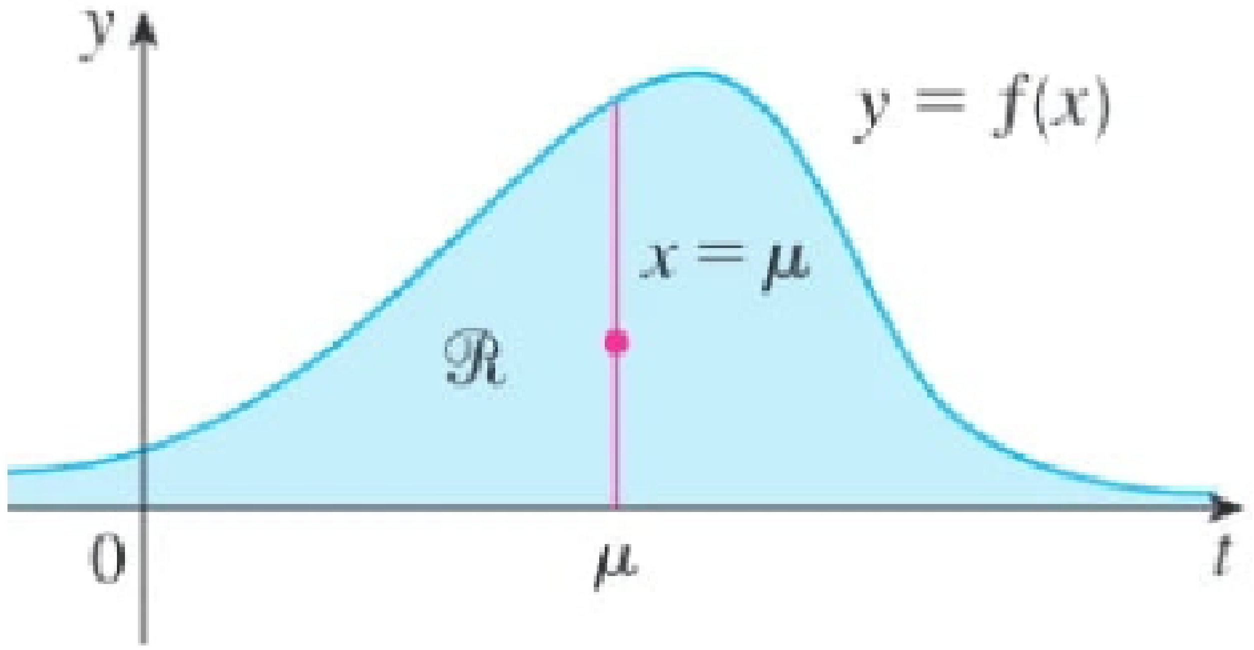


FIGURE 4
$\mathcal{R}$ balances at a point on the line $x = \mu$

EXAMPLE 3 Find the mean of the exponential distribution of Example 2:

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ ce^{-ct} & \text{if } t \geqslant 0 \end{cases}$$

$u = t \qquad v = -e^{-ct}$

$du = dt \qquad dv = ce^{-ct}\,dt$

$$\int_0^\infty t\left(ce^{-ct}\right)dt = \mu$$

$$\lim_{x\to\infty}\left[-te^{-ct} + \int_0^x e^{-ct}\,dt\right]$$

$$c = \frac{1}{\mu} = \mu^{-1}$$

$$\lim_{x\to\infty}\left[-te^{-ct} + \left(-\frac{1}{c}e^{-ct}\right)\right]_0^x$$

$$\lim_{x\to\infty}\left(-xe^{-xt} - \frac{1}{c}e^{-cx} + \frac{1}{c}\right) = \frac{1}{c} = \mu$$

EXAMPLE 3 Find the mean of the exponential distribution of Example 2:

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ ce^{-ct} & \text{if } t \geqslant 0 \end{cases}$$

$f(t) = \mu^{-1} e^{-t/\mu}$ ← mean

$f(t) = c e^{-ct}$ ← starting point

$f(t) = A e^{-ct}$ ← basic exponetial function

EXAMPLE 4 Suppose the average waiting time for a customer's call to be answered by a company representative is five minutes.
(a) Find the probability that a call is answered during the first minute, assuming that an exponential distribution is appropriate.
(b) Find the probability that a customer waits more than five minutes to be answered.

$$f(t) = \mu^{-1} e^{-t/\mu}$$

$$f(t) = 0.2 e^{-t/5}$$

$$\int_0^1 0.2 e^{-t/5}\, dt = (0.2)(-5)e^{-t/5}\Big]_0^1$$

$$= -1e^{-1/5} + 1$$

$$= 1 - e^{-1/5} = 0.181$$

18%

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(b) Find the probability that a customer waits more than five minutes to be answered.

$$f(t) = \mu^{-1} e^{-t/\mu}$$

$$f(t) = 0.2 e^{-t/5}$$

$$\int_5^\infty 0.2 e^{-t/5}\, dt = (0.2)(-5)e^{-t/5}\Big]_5^\infty$$

$$= 0 + e^{-1} = 0.3679$$

$$= 36.8\%$$

$$\int e^{c\boxed{x}}$$

$$u = cx$$

$$du = c\,dx$$

$$\frac{1}{c}\,du = dx$$

$$\frac{1}{c}\int e^{u}\,du$$

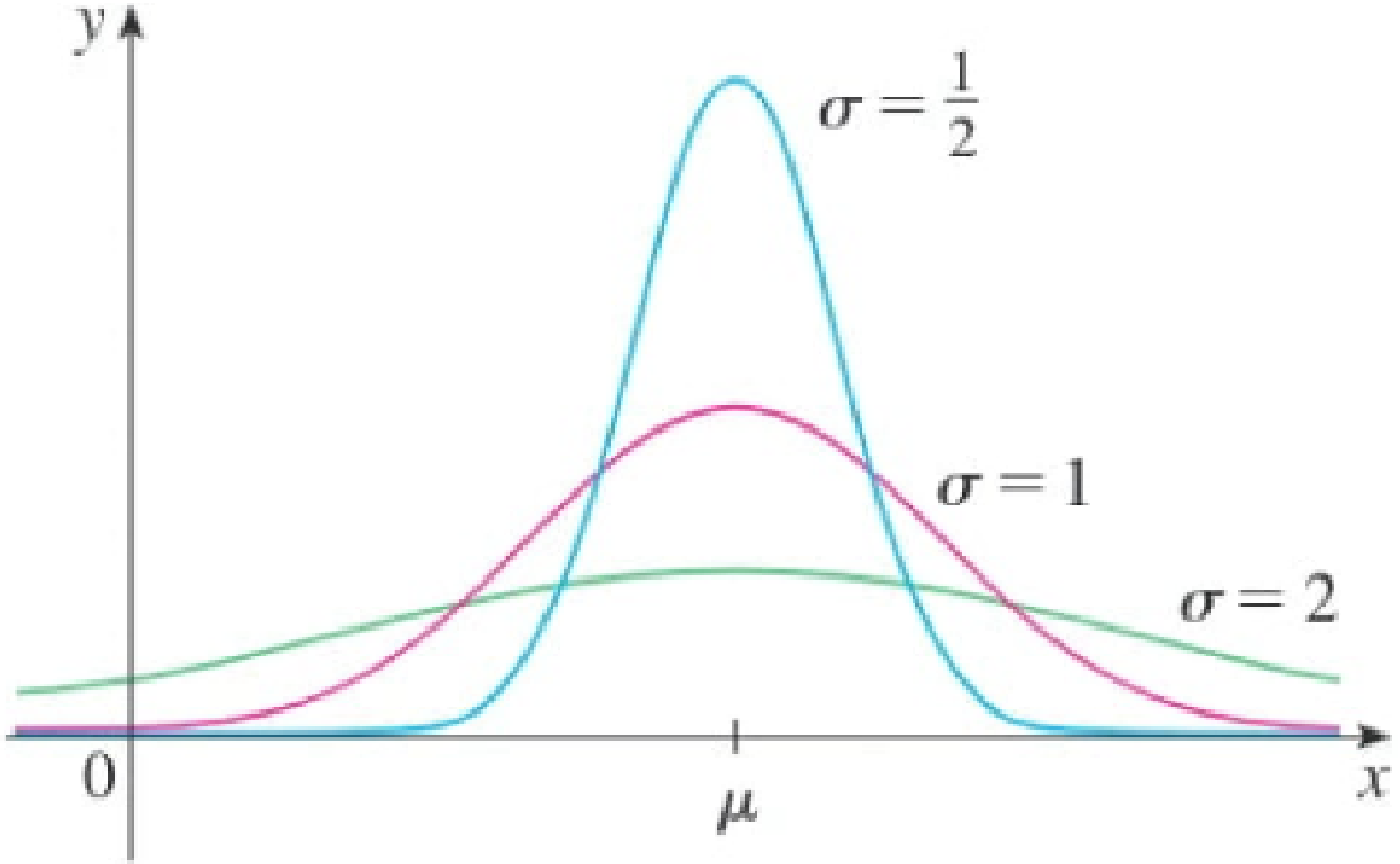
$$\frac{1}{c}e^{u} = \frac{1}{c}e^{cx}$$

NormalDistributions

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}}$$

μ = mean

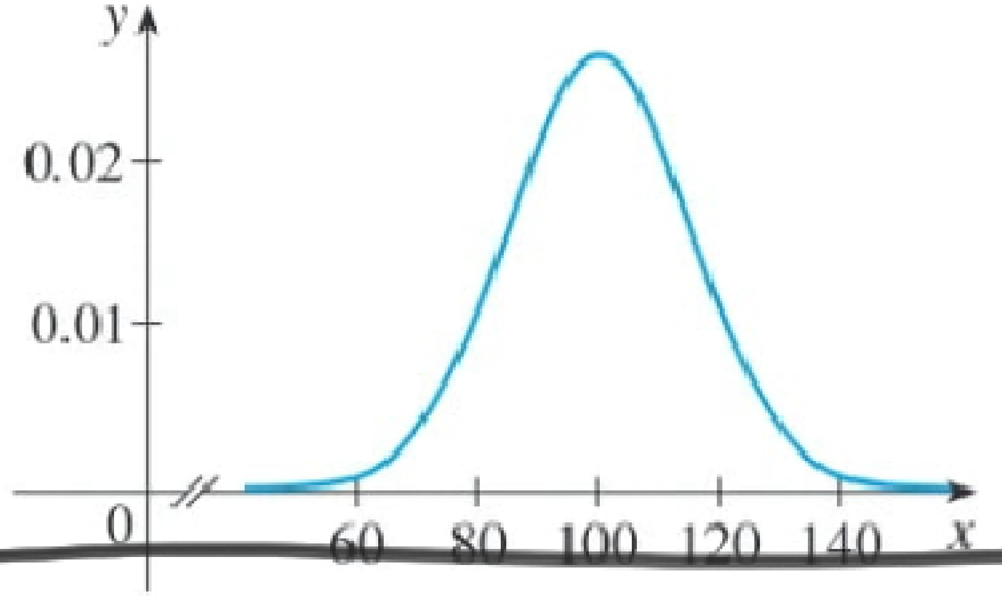
σ = standard Deviation



EXAMPLE 5 Intelligence Quotient (IQ) scores are distributed normally with mean 100 and standard deviation 15. (Figure 6 shows the corresponding probability density function.)

(a) What percentage of the population has an IQ score between 85 and 115?

(b) What percentage of the population has an IQ above 140?



$$f(x) = \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2\cdot 15^2}}$$

$$\int_{85}^{115} \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2\cdot 15^2}}\,dx$$

$$\boxed{0.68 \text{ or } 68\%}$$

1	2	3
68	95	99.9

68-95-99.9

85 ⟷ 115

1 st. D.

(a) A type of light bulb is labeled as having an average lifetime of 1000 hours. It's reasonable to model the probability of failure of these bulbs by an exponential density function with mean $\mu = 1000$. Use this model to find the probability that a bulb
 (i) fails within the first 200 hours,
 (ii) burns for more than 800 hours.

(b) What is the median lifetime of these light bulbs?

$$f(t) = \mu^{-1} e^{-t/\mu}$$

$$f(t) = 0.001\, e^{-t/1000}$$

$$-e^{-t/1000}$$

$$\int 0.001\, e^{-t/1000}\, dt$$

$$(0.001)(-1000)\, e^{-t/1000}$$

$$-e^{-1/5} + 1 \longrightarrow \text{i) } 0 \to 200 \quad 0.18127$$

$$0 + e^{-4/5} \longrightarrow \text{ii) } 800 \to \infty \quad 0.4493$$

Boxes are labeled as containing 500 g of cereal. The machine filling the boxes produces weights that are normally distributed with standard deviation 12 g.

(a) If the target weight is 500 g, what is the probability that the machine produces a box with less than 480 g of cereal?

(b) Suppose a law states that no more than 5% of a manufacturer's cereal boxes can contain less than the stated weight of 500 g. At what target weight should the manufacturer set its filling machine?

$$\int \frac{1}{12\sqrt{2\pi}} e^{\left(-\frac{(x-500)^2}{2\cdot 12^2}\right)}$$

$$\int_0^{480} \rightarrow 4.78\%$$