Chapter 9
Differential Equations

9.3 Separable Equations

Separable Equations (1 of 4)

A **separable equation** is a first-order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of x times a function of y.

In other words, it can be written in the form

$$\frac{dy}{dx} = g(x)f(y)$$

The name *separable* comes from the fact that the expression on the right side can be "separated" into a function of x and a function of y.

Separable Equations (2 of 4)

Equivalently, if $f(y) \neq 0$, we could write

$$1 \quad \frac{dy}{dx} = \frac{g(x)}{h(y)}$$

where $h(y) = \dfrac{1}{f(y)}$.

To solve this equation we rewrite it in the differential form

$$h(y)\,dy = g(x)\,dx$$

so that all y's are on one side of the equation and all x's are on the other side.

Separable Equations (3 of 4)

Then we integrate both sides of the equation:

$$2 \quad \int h(y)\,dy = \int g(x)\,dx$$

Equation 2 defines y implicitly as a function of x. In some cases we may be able to solve for y in terms of x.

Separable Equations (4 of 4)

We use the Chain Rule to justify this procedure: If h and g satisfy (2), then

$$\frac{d}{dx}\left(\int h(y)\,dy\right) = \frac{d}{dx}\left(\int g(x)\,dx\right)$$

so

$$\frac{d}{dy}\left(\int h(y)\,dy\right)\frac{dy}{dx} = g(x)$$

and

$$h(y)\frac{dy}{dx} = g(x)$$

Thus Equation 1 is satisfied.

Example 1

(a) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$.

(b) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

Solution:

(a) We write the equation in terms of differentials and integrate both sides:

$$y^2 \, dy = x^2 \, dx$$
$$\int y^2 \, dy = \int x^2 \, dx$$

Example 1 – *Solution* (1 of 2)

$$\frac{1}{3}y^3 = \frac{1}{3}x^3 + C$$

where C is an arbitrary constant.

(We could have used a constant C_1 on the left side and another constant C_2 on the right side. But then we could combine these constants by writing $C = C_2 - C_1$.)

Solving for y, we get

$$y = \sqrt[3]{x^3 + 3C}$$

Example 1 – *Solution* (2 of 2)

We could leave the solution like this or we could write it in the form

$$y = \sqrt[3]{x^3 + K}$$

where $K = 3C$. (Since C is an arbitrary constant, so is K.)

(b) If we put $x = 0$ in the general solution in part (a), we get $y(0) = \sqrt[3]{K}$.

To satisfy the initial condition $y(0) = 2$, we must have $\sqrt[3]{K} = 2$ and so $K = 8$.

Thus the solution of the initial-value problem is

$$y = \sqrt[3]{x^3 + 8}$$

Orthogonal Trajectories

Orthogonal Trajectories (1 of 2)

An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family orthogonally, that is, at right angles (see Figure 7).
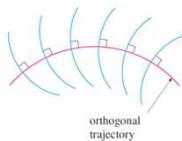

Figure 7

Orthogonal Trajectories (2 of 2)

For instance, each member of the family $y = mx$ of straight lines through the origin is an orthogonal trajectory of the family $x^2 + y^2 = r^2$ of concentric circles with center the origin (see Figure 8). We say that the two families are orthogonal trajectories of each other.
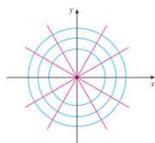

Figure 8

Example 5

Find the orthogonal trajectories of the family of curves $x = ky^2$, where is k an arbitrary constant.

Solution:

The curves $x = ky^2$ form a family of parabolas whose axis of symmetry is the x-axis.

The first step is to find a single differential equation that is satisfied by all members of the family.

If we differentiate $x = ky^2$, we get

$$1 = 2ky \frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{2ky}$$

Example 5 – *Solution* (1 of 3)

This differential equation depends on k, but we need an equation that is valid for all values of k simultaneously.

To eliminate k we note that, from the equation of the given general parabola $x = ky^2$, we have $k = \frac{x}{y^2}$ and so the differential equation can be written as

$$\frac{dy}{dx} = \frac{1}{2ky} = \frac{1}{2\frac{x}{y^2} y}$$

or

$$\frac{dy}{dx} = \frac{y}{2x}$$

Example 5 – *Solution* (2 of 3)

This means that the slope of the tangent line at any point (x, y) on one of the parabolas is $y' = \frac{y}{(2x)}$.

On an orthogonal trajectory the slope of the tangent line must be the negative reciprocal of this slope.

Therefore the orthogonal trajectories must satisfy the differential equation

$$\frac{dy}{dx} = -\frac{2x}{y}$$

This differential equation is separable, and we solve it as follows:

$$\int y \, dy = -\int 2x \, dx$$

Example 5 – *Solution* (3 of 3)

$$\frac{y^2}{2} = -x^2 + C$$

$$4 \quad x^2 + \frac{y^2}{2} = C$$

where C is an arbitrary positive constant.

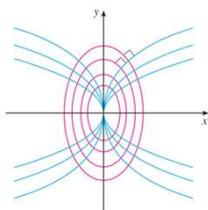Thus the orthogonal trajectories are the family of ellipses given by Equation 4 and sketched in Figure 9.

Figure 9

Mixing Problems

Mixing Problems (1 of 2)

A typical mixing problem involves a tank of fixed capacity filled with a thoroughly mixed solution of some substance, such as salt.

A solution of a given concentration enters the tank at a fixed rate and the mixture, thoroughly stirred, leaves at a fixed rate, which may differ from the entering rate.

If $y(t)$ denotes the amount of substance in the tank at time t, then $y'(t)$ is the rate at which the substance is being added minus the rate at which it is being removed.

Mixing Problems (2 of 2)

The mathematical description of this situation often leads to a first-order separable differential equation.

We can use the same type of reasoning to model a variety of phenomena: chemical reactions, discharge of pollutants into a lake, injection of a drug into the bloodstream.

Example 6

A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?

Solution:

Let $y(t)$ be the amount of salt (in kilograms) after t minutes.

We are given that $y(0) = 20$ and we want to find $y(30)$. We do this by finding a differential equation satisfied by $y(t)$.

Example 6 – *Solution* (1 of 4)

Note that $\frac{dy}{dt}$ is the rate of change of the amount of salt, so

$$5 \quad \frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

where (rate in) is the rate at which salt enters the tank and (rate out) is the rate at which salt leaves the tank.

We have

$$\text{rate in} = \left(0.03\,\frac{kg}{L}\right)\left(25\,\frac{L}{min}\right) = 0.75\,\frac{kg}{min}$$

The tank always contains 5000 L of liquid, so the concentration at time t is $\frac{y(t)}{5000}$

(measured in kilograms per liter).

Example 6 – *Solution* (2 of 4)

Since the brine flows out at a rate of 25 L/min, we have

$$\text{rate out} = \left(\frac{y(t)\ \text{kg}}{5000\ \text{L}}\right)\left(25\frac{\text{L}}{\text{min}}\right) = \frac{y(t)\ \text{kg}}{200\ \text{min}}$$

Thus, from Equation 5, we get

$$\frac{dy}{dt} = 0.75 - \frac{y(t)}{200} = \frac{150 - y(t)}{200}$$

Solving this separable differential equation, we obtain

$$\int \frac{dy}{150 - y} = \int \frac{dt}{200}$$

$$-\ln|150 - y| = \frac{t}{200} + C$$

Example 6 – *Solution* (3 of 4)

Since $y(0) = 20$, we have $-\ln 130 = C$, so

$$-\ln|150 - y| = \frac{t}{200} - \ln 130$$

Therefore

$$|150 - y| = 130e^{-t/200}$$

Since $y(t)$ is continuous and $y(0) = 20$ and the right side is never 0, we deduce that $150 - y(t)$ is always positive.

Example 6 – *Solution* (4 of 4)

Thus $|150 - y| = 150 - y$ and so

$$y(t) = 150 - 130e^{-t/200}$$

The amount of salt after 30 min is

$$y(30) = 150 - 130e^{-30/200}$$
$$\approx 38.1\ \text{kg}$$