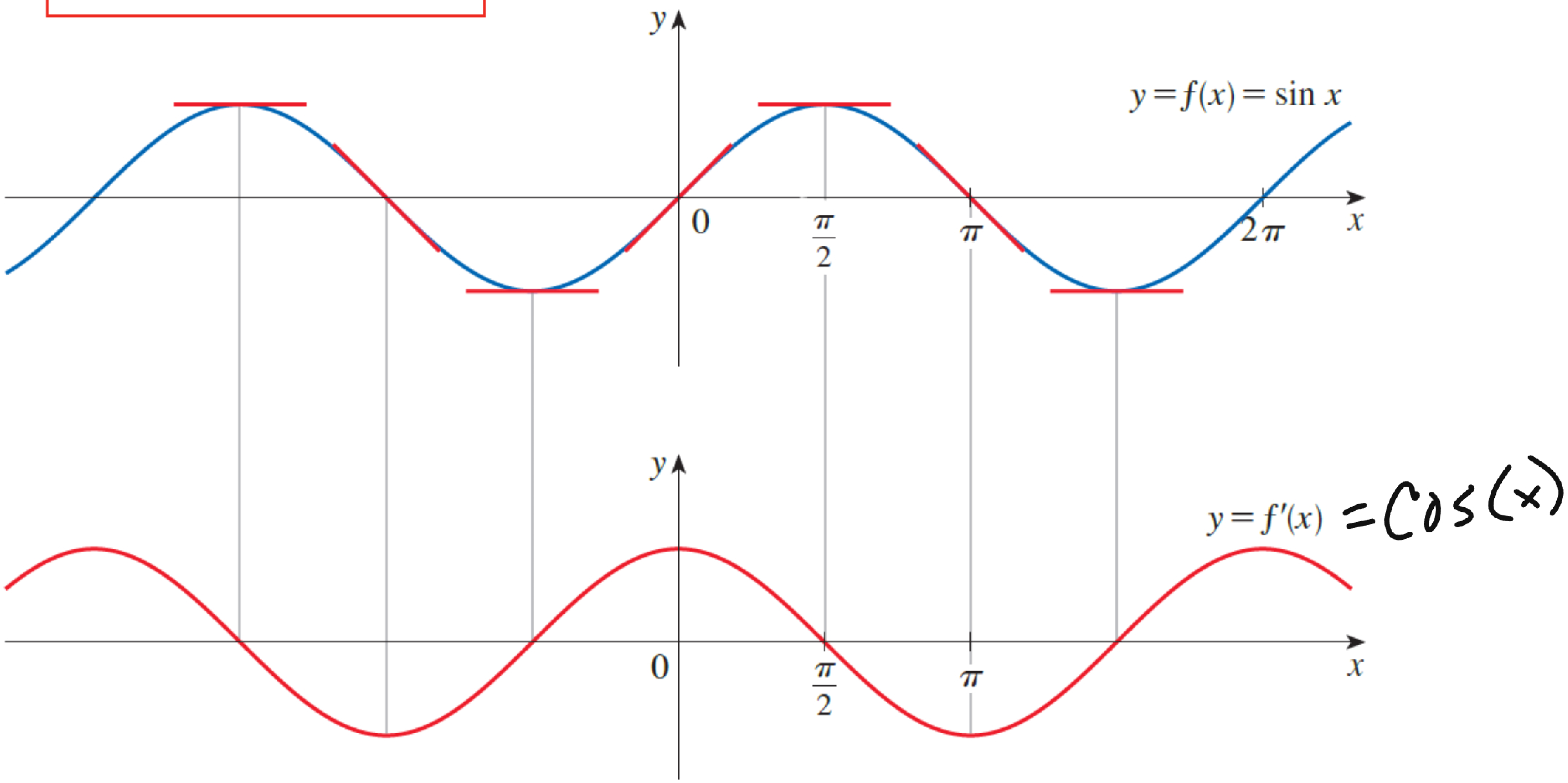
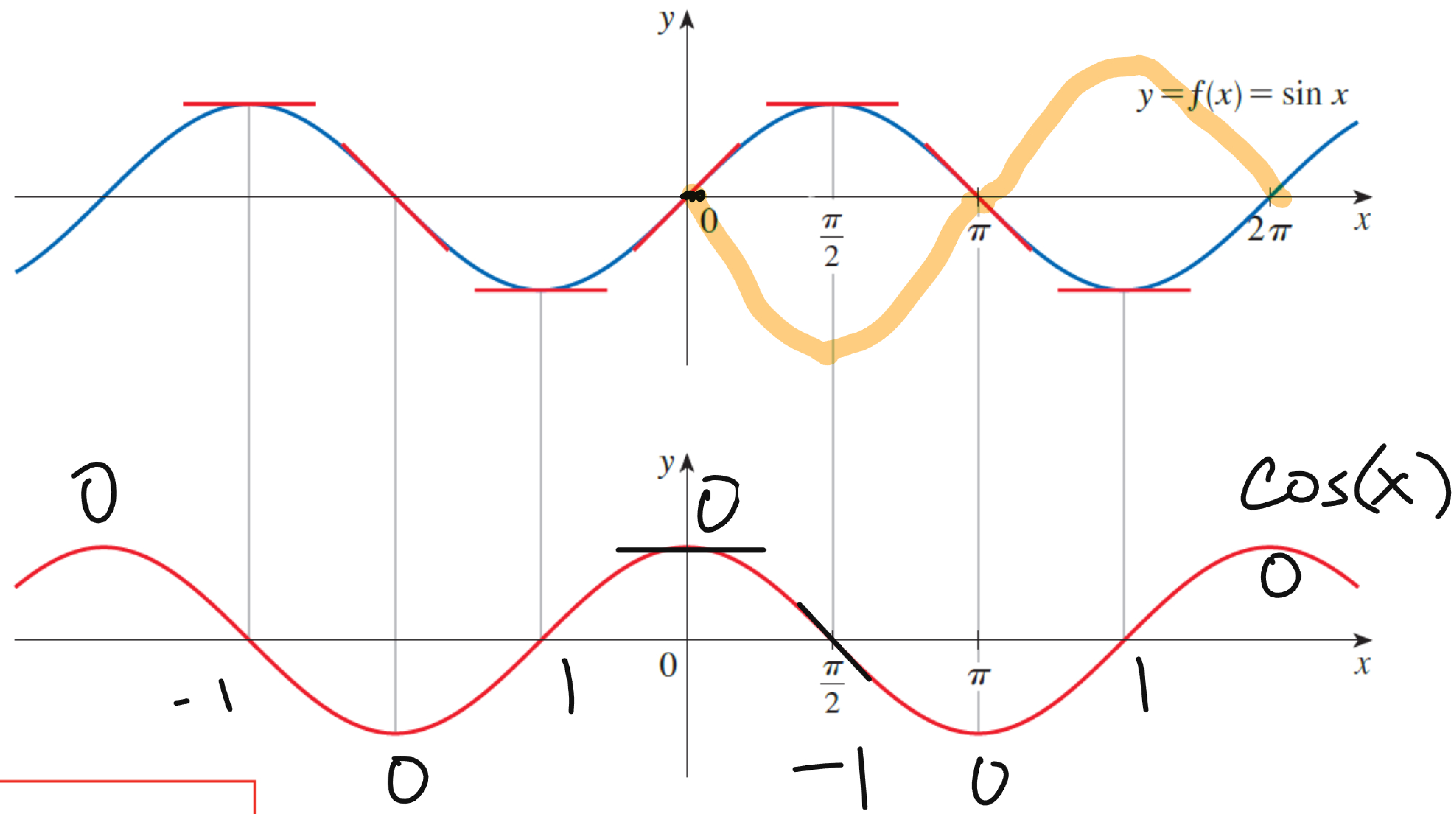
3.3 Derivatives of Trigonometric Functions

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$





$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

$$\frac{d}{dx}(\tan x)$$

$f \to \sin x \quad f' \to \cos x$

$g \to \cos x \quad g' \to -\sin x$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\frac{f'g - fg'}{g^2}$$

$$\frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$

$$\frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

$$\sin^2\theta = 1 - \cos^2\theta \qquad \sec^2\theta - 1 = \tan^2\theta \qquad \csc^2\theta - 1 = \cot^2\theta$$

$$\cos^2\theta = 1 - \sin^2\theta \qquad 1 = \sec^2\theta - \tan^2\theta \qquad 1 = \csc^2\theta - \cot^2\theta$$

Double - angle formulas :

$$\sin 2\theta = 2 \cdot \sin\theta \cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

EXAMPLE 3 An object fastened to the end of a vertical spring is stretched 4 cm beyond its rest position and released at time $t = 0$. (See Figure 4 and note that the downward direction is positive.) Its position at time t is

$$s = f(t) = 4 \cos t$$

Find the velocity and acceleration at time t and use them to analyze the motion of the object.

$$f'(t) = 4(-\sin t) = -4 \sin t$$

$$f''(t) = -4 \cos t$$

$$f(w) = \frac{1 + \sec w}{1 - \sec w}$$

$f \to 1 + \sec w \quad f' \to \sec w \tan w$

$g \to 1 - \sec w \quad g' \to -\sec w \tan w$

$$f'(w) = \frac{(\sec w \tan w)(1 - \sec w) - (1 + \sec w)(-\sec w \tan w)}{(1 - \sec w)^2}$$

$$\to \sec w \tan w - \sec^2 w \tan w + \sec w \tan w + \sec^2 w \tan w$$

$$= \boxed{\frac{2 \sec w \tan w}{(1 - \sec w)^2}}$$

Differentiate $y = \sin^2 x. = (\sin x)^2$

$\sin^2 x \rightarrow (\sin x)(\sin x)$

$(\cos x)(\sin x) + (\sin x)(\cos x)$

$\boxed{2 \cos x \sin x}$

$(\sin x)^2 \rightarrow 2 \cos x \sin x$

3.4 The Chain Rule

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x) \tag{1}$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \tag{2}$$

$\sin^2 x \rightarrow (\sin x)^2$

u^2

$u = \sin x$

$u' = \cos x$

$2u$

$\sin^2 x \rightarrow 2(\sin x)\cos x$

$= 2\sin x \cos x$

$$\sin(x^2)$$

$$u = x^2$$

$$u' = 2x$$

$$\sin(u)$$

$$\downarrow$$

$$\cos(u)$$

$$\sin(x^2) \longrightarrow \cos(x^2)(2x)$$

$$= 2x\cos(x^2)$$

$(f \circ g)(x)$

Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$.

$$f(x) = \sqrt{x^2+1}$$

$$f'(x) = \underbrace{(2x)}_{u'}\underbrace{\left(\frac{1}{2\sqrt{x^2+1}}\right)}_{f'(u)}$$

$$= \boxed{\frac{x}{\sqrt{x^2+1}}}$$

$$u = x^2+1$$

$$u' = 2x$$

$$f(u) = u^{1/2}$$

$$f'(u) = \frac{1}{2}u^{-1/2}$$

$$= \frac{1}{2\sqrt{u}}$$

Find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}} = (x^2+x+1)^{-1/3}$

$u = x^2 + x + 1$

$u' = 2x + 1$

$f(u) = u^{-1/3}$

$f'(u) = -\frac{1}{3} u^{-4/3}$

$= \dfrac{-1}{3u^{4/3}}$

$f'(x) =$

$(2x+1)\left(\dfrac{-1}{3(x^2+x+1)^{4/3}}\right)$

$$\boxed{f'(x) = \frac{-(2x+1)}{3(x^2+x+1)^{4/3}}}$$

Differentiate $y = e^{\sin x}$

$$y' = (\cos x)(e^{\sin x})$$

$$u = \sin x$$

$$u' = \cos x$$

$$e^{u}$$

$$e^{u}$$

$$y = (2x+1)^5(x^3 - x + 1)^4$$

$$f = (2x+1)^5 \rightarrow \begin{array}{l} u = 2x+1 \\ u' = 2 \\ f(u) = u^5 \\ f'(u) = 5u^4 \end{array}$$

$$f' = (2)(5(2x+1)^4)$$

$$g = (x^3 - x + 1)^4$$

$$g' = (3x^2-1)(4(x^3-x+1)^3)$$

$$y' = 10(2x+1)^4(x^3-x+1)^4 + (12x^2-4)(x^3-x+1)^3(2x+1)^5$$

$f(x) = \sin(\cos(\tan x))$

$U = \cos(\tan x)$

$U' = (\sec^2 x)(-\sin(\tan x))$

$\cancel{f}(u) = \sin(u)$

$f'(u) = \cos U$

$w = t$

$w' = s$

$U(w) =$

$U'(w) =$

$f'(x) = -\sec^2 x(\sin(\tan x))(\cos(\cos(\tan x)))$

$$e^{x\ln b} \rightarrow$$

$$(\ln b)\boxed{\left(e^{x\ln b}\right)}$$

$$u = x\ln b$$

$$u' = \ln b$$

$$f(u) = e^{u}$$

$$f'(u) = e^{u}$$

$$\frac{d}{dx}\left(b^{x}\right) = (\ln b)\left(b^{x}\right)$$

$$g(x) = 2^x$$

$$(\ln b)(b^x)$$

$$g'(x) = (\ln 2)(2^x)$$

$$h(x) = e^x$$

$$h'(x) = (\ln e)(e^x) = e^x$$

$$y = \left(x + \frac{1}{x}\right)^5$$

$$u = x + x^{-1}$$

$$u' = 1 - x^{-2} = 1 - \frac{1}{x^2}$$

$$y' = \left(1 - \frac{1}{x^2}\right)\left(5\left(x + 1/x\right)^4\right)$$

$$y \to u^5 \to 5u^4$$

$$f \to x^2 e^x \qquad f' = (2x)(e^x) + (x^2)(e^x)$$

$$2x \qquad e^x$$

$$g \to x^2 + e^x \qquad g' = 2x + e^x$$

$$\frac{(2xe^x + x^2e^x)(x^2+e^x) - (x^2e^x)(2x+e^x)}{(x^2+e^x)^2}$$