1 Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

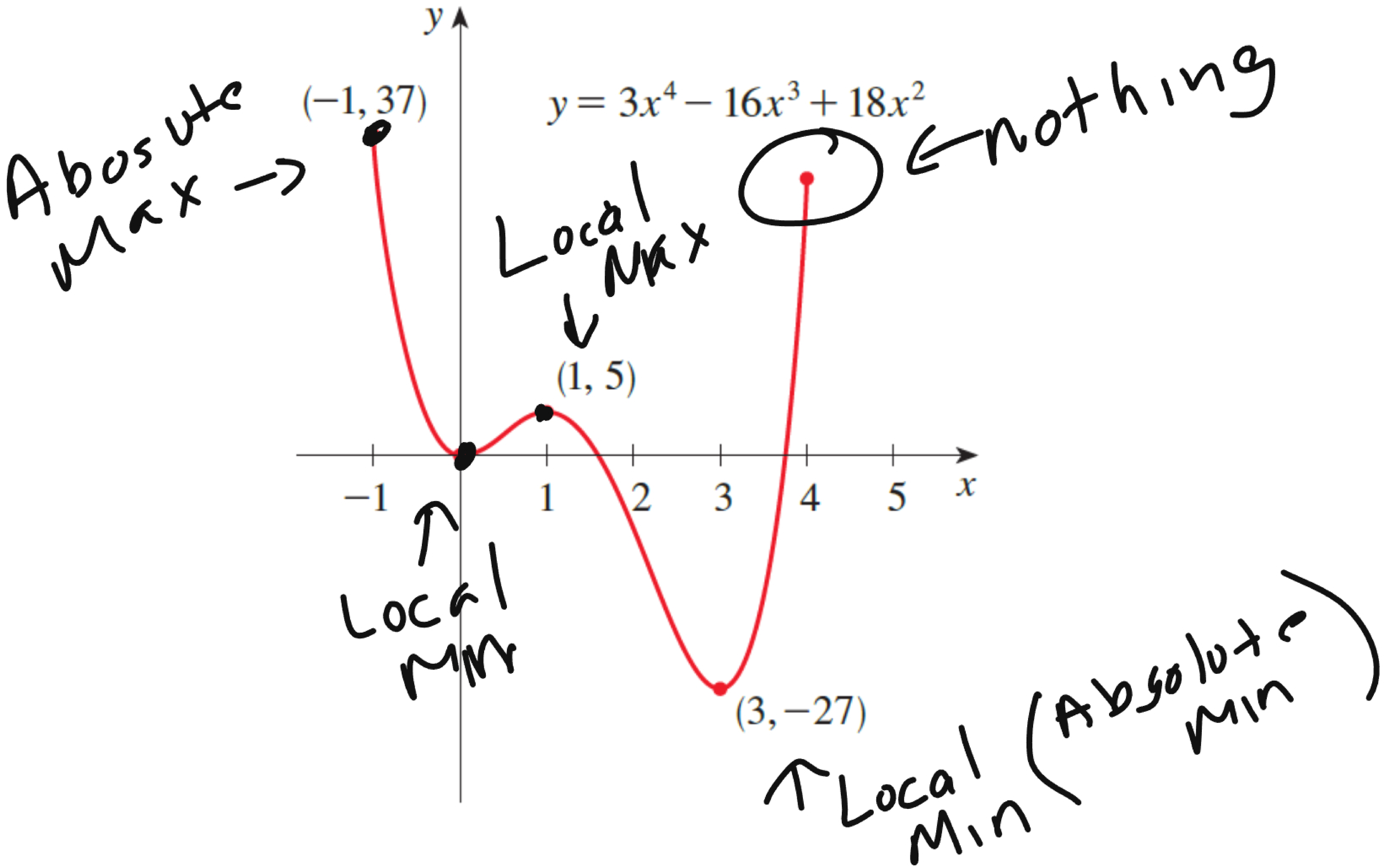
- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

2 Definition The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geqslant f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leqslant f(x)$ when x is near c.

3 The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

$$f(x) = 3x^4 - 16x^3 + 18x^2 \qquad -1 \leqslant x \leqslant 4$$



4 Fermat's Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

6 Definition A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

$$f(x) = x^3$$

$$f'(x) = 3x^2 = 0$$

critical number $c = 0$

EXAMPLE 10 The Hubble Space Telescope was deployed on April 24, 1990, by the space shuttle *Discovery*. A model for the velocity of the shuttle during this mission, from liftoff at $t = 0$ until the solid rocket boosters were jettisoned at $t = 126$ seconds, is given by

$$v(t) = 0.001302t^3 - 0.09029t^2 + 23.61t - 3.083$$

(in feet per second). Using this model, estimate the absolute maximum and minimum values of the *acceleration* of the shuttle between liftoff and the jettisoning of the boosters.

$[0, 126]$

$c = 0$

$c = 126$

$a(t) = v'(t) = 0.003906t^2 - 0.18058t +$

$a'(t) = 0.007812t - 0.18058$

$0 = 0.007812t - 0.18058$

$= 23.12 \approx t$

x	$0.003906x^2 - 0.18058x + 23.61$
0	23.61
23.12	21.522882
126	62.868576

40. $g(x) = \sqrt[3]{4 - x^2}$

$c = 0, -2, 2$

$u = 4 - x^2 \rightarrow u' = -2x$

$f(u) = u^{1/3} \rightarrow f'(u) = \frac{1}{3}u^{-2/3} \cdot u'$

$g'(x) = \frac{-2x}{3(4-x^2)^{2/3}}$

$3(4-x^2)^{2/3} = 0$

$4 - x^2 = 0$

$-x^2 = -4 \rightarrow x^2 = 4$

$x = \pm 2$

$$C'(x) = 0.135\left[1(e \quad) + x(\quad)\right]$$

$$C'(x) = 0.135(e^{-2.802x})(1-2.802x)$$

$$0 = \cancel{0.135}\cancel{(e^{-2.802x})}\boxed{(1-2.802x)}$$

$$0 = 1 - 2.802x$$

$$-1 = -2.802x$$

$$1/2.802 = x \approx .36 \text{ hours}$$

After an antibiotic tablet is taken, the concentration of the antibiotic in the bloodstream is modeled by the function

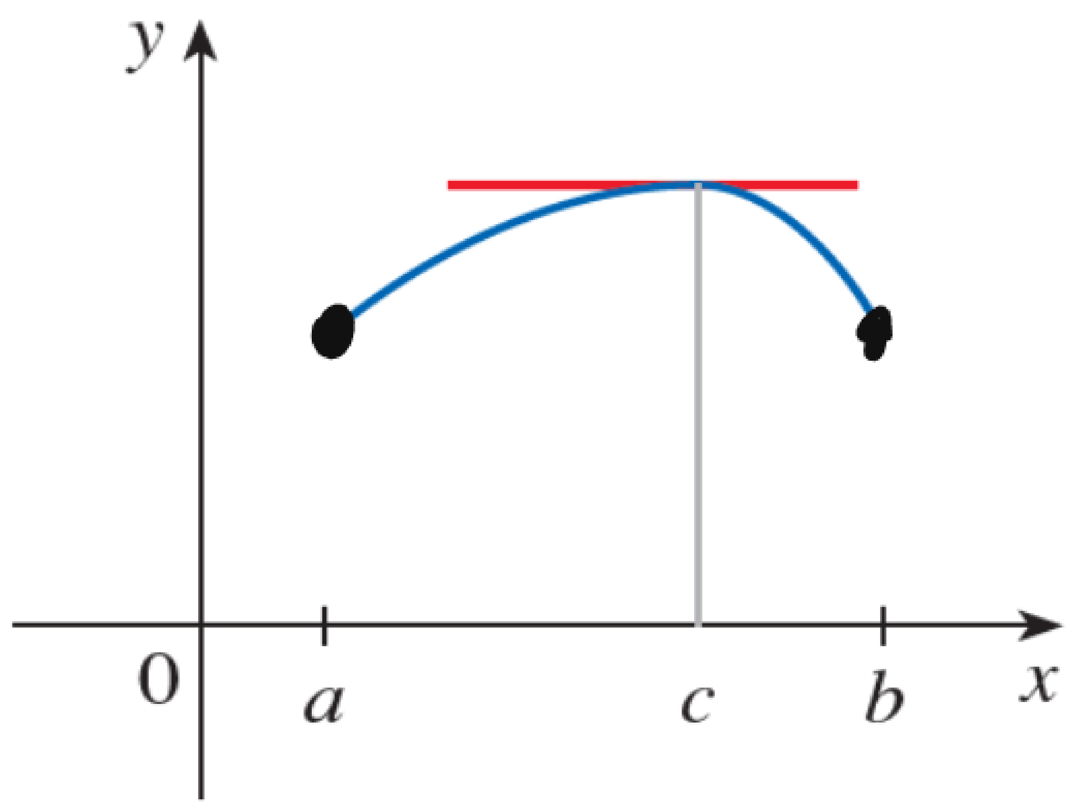
$$C(t) = 8(e^{-0.4t} - e^{-0.6t})$$

where the time t is measured in hours and C is measured in μg/mL. What is the maximum concentration of the antibiotic during the first 12 hours?

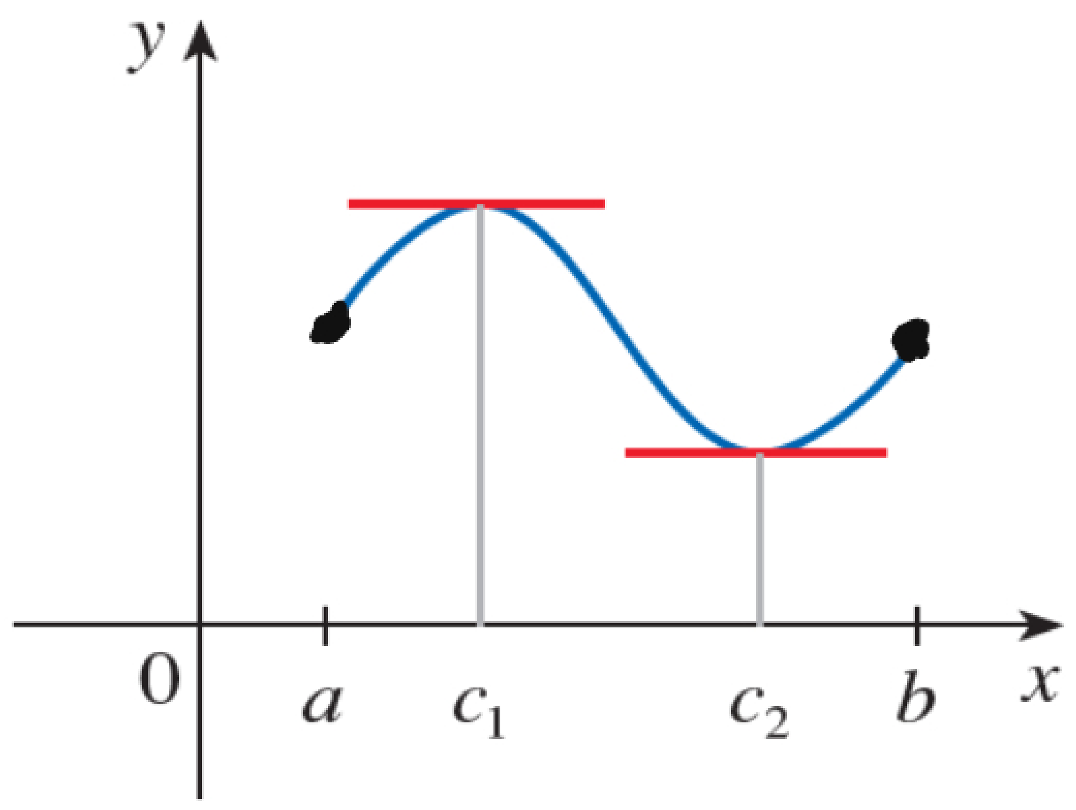
Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

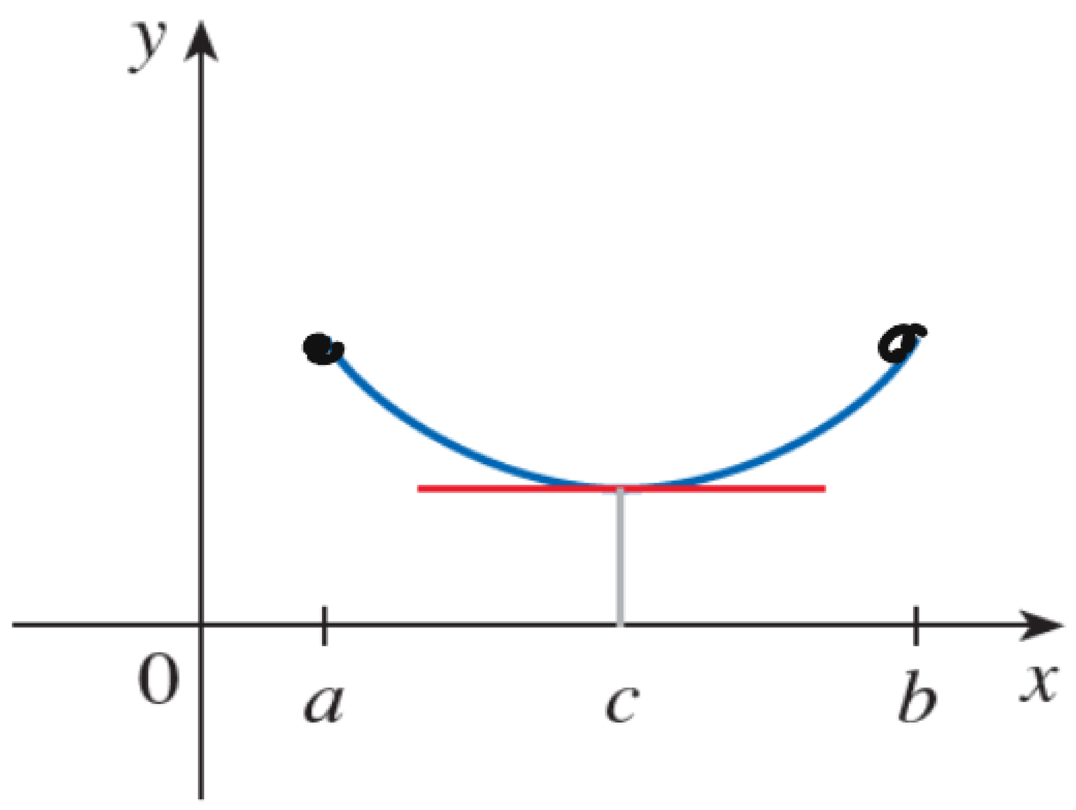
Then there is a number c in (a, b) such that $f'(c) = 0$.



(b)



(c)



(d)

EXAMPLE 2 Prove that the equation $x^3 + x - 1 = 0$ has exactly one real solution.

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

$(0, -1)$ $(1, 1)$.

$f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1 > 0$

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$\boxed{1} \qquad f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$\boxed{2} \qquad f(b) - f(a) = f'(c)(b - a)$$

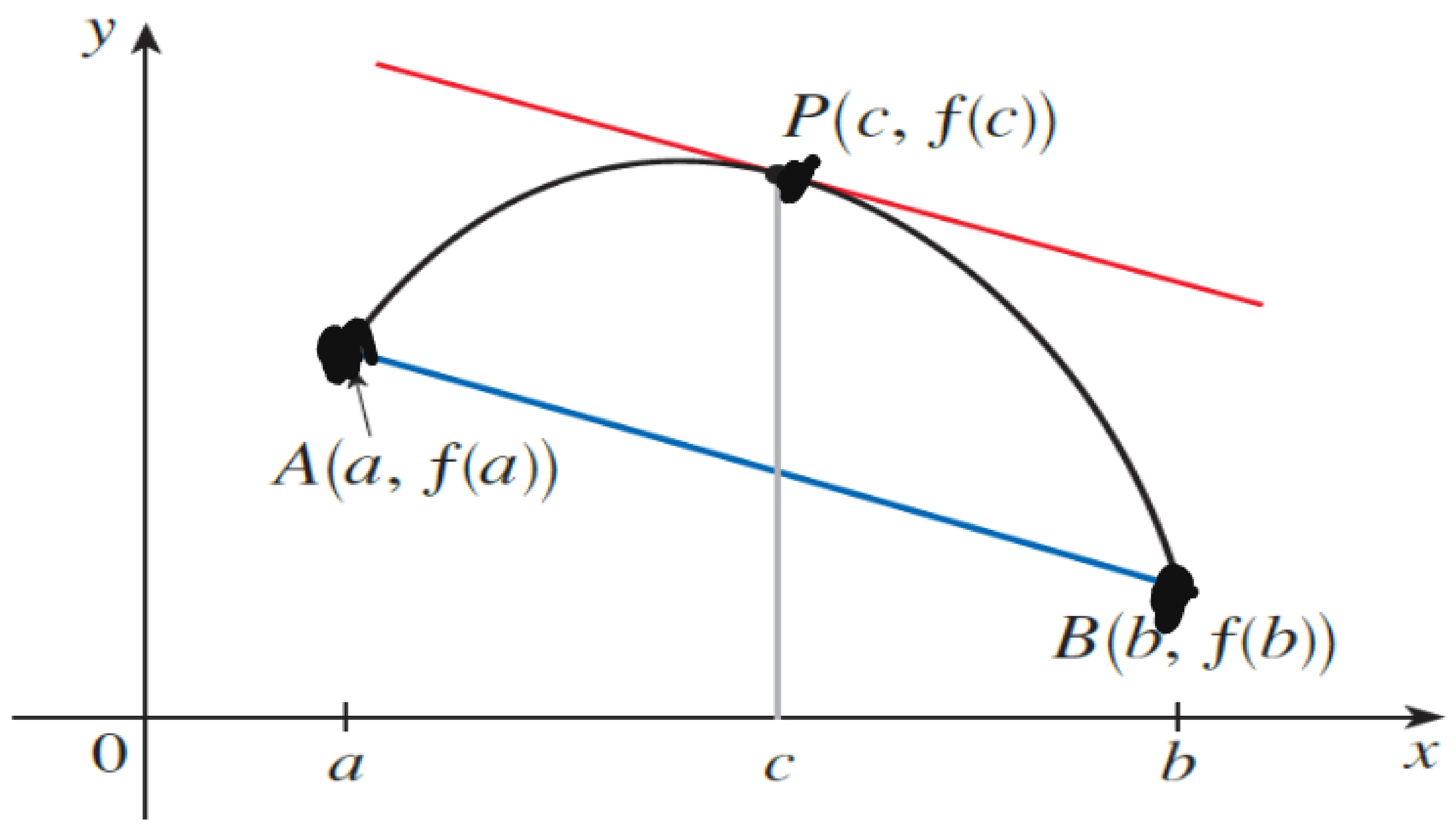


FIGURE 3

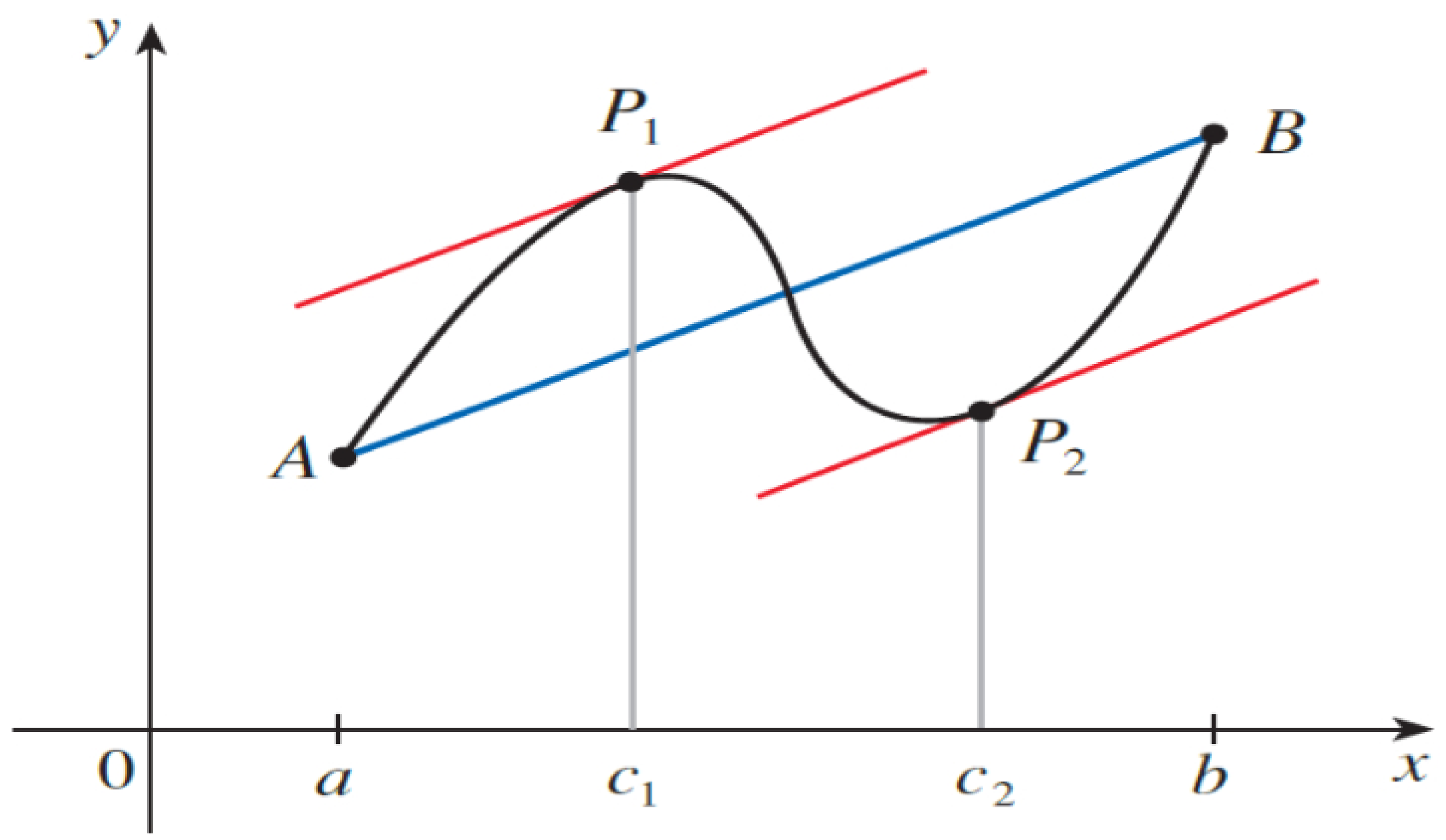


FIGURE 4

EXAMPLE 4 If an object moves in a straight line with position function $s = f(t)$, then the average velocity between $t = a$ and $t = b$ is

$$\frac{f(b) - f(a)}{b - a}$$

and the velocity at $t = c$ is $f'(c)$. Thus the Mean Value Theorem (in the form of Equation 1) tells us that at some time $t = c$ between a and b the instantaneous velocity $f'(c)$ is equal to that average velocity. For instance, if a car traveled 180 km in 2 hours, then the speedometer must have read 90 km/h at least once.

EXAMPLE 5 Suppose that $f(0) = -3$ and $f'(x) \leqslant 5$ for all values of x. How large can $f(2)$ possibly be?

$f'(x)$ max $= 5$ slope

$0 \to 2$ $\qquad f(2) = 7$

$-3 \to 7$

$f(0) = -3$ $\quad +10$

$f(2) = 7$

a number c such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

so
$$f(2) = f(0) + 2f'(c) = -3 + 2f'(c)$$

We are given that $f'(x) \leqslant 5$ for all x, so in particular we know that $f'(c) \leqslant 5$. Multiply-

$$f(x) = x^3 - 3x + 2, \quad \boxed{[-2, 2]}$$

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

1 $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

2 $$f(b) - f(a) = f'(c)(b - a)$$

$$f'(c) = \frac{4 - 0}{2 + 2} = \frac{4}{4} = 1$$

$$f'(x) = 3x^2 - 3$$

$$3c^2 - 3 = 1$$

$$c^2 = \frac{4}{3} \rightarrow \boxed{c = \pm\frac{2}{\sqrt{3}}}$$

$$f(x) = \ln x, \quad [1, 4]$$

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.

2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a) \tag{2}$$

$$f'(x) = \frac{1}{x}$$

$$f(1) = \ln(1) = 0$$

$$f(4) = \ln(4)$$

$$f'(c) = \frac{\ln(4) - 0}{4 - 1} = \frac{\ln 4}{3}$$

$$\frac{1}{c} = \frac{\ln 4}{3} \rightarrow \boxed{c = \frac{3}{\ln 4}}$$