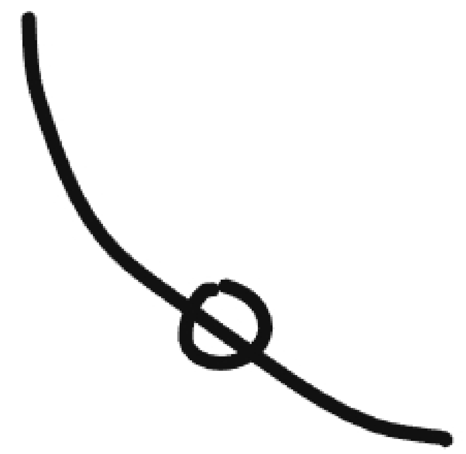
$$\lim_{x\to 1}\frac{x^2-x}{x^2-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$= \frac{x\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} = \frac{x}{x+1} = \boxed{\frac{1}{2}}$$



$$\lim_{x\to\infty}\frac{x^2-1}{2x^2+1} \to \frac{\infty}{\infty}$$

$$\frac{x^2/x^2 - \frac{1}{x^2}}{2x^2/x^2 + 1/x^2} = \frac{1 - 1/x^2}{2 + 1/x^2} = \boxed{\frac{1}{2}}$$

Horizontal Asy

Indeterminate Forms and l'Hospital's Rule

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x - 1}$$

$$\lim_{x \to 1} \frac{\ln x}{x - 1} \to \frac{0}{0}$$

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)}} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} \overset{H}{=} \frac{2x - 1}{2x} = \frac{1}{2}$$

EXAMPLE 1 Find $\lim_{x \to 1} \frac{\ln x}{x - 1}$. $= \frac{0}{0}$ Indeterminate form

$f(x) = \ln x \qquad f'(x) = 1/x$

$g(x) = x - 1 \qquad g'(x) = 1$

$$\lim_{x \to 1} \frac{1/x}{1} = 1$$

EXAMPLE 2 Calculate $\lim_{x \to \infty} \frac{e^x}{x^2} \to \frac{\infty}{\infty}$

$f(x) = e^x$ $\quad f'(x) = e^x$ $\quad f''(x) = e^x$

$g(x) = x^2$ $\quad g'(x) = 2x$ $\quad g''(x) = 2$

$$\lim_{x \to \infty} \frac{e^x}{2x} = \frac{\infty}{\infty}$$

$$\lim_{x \to \infty} \frac{e^x}{2} = \frac{\infty}{2} = \infty$$

$$\lim_{x \to 0} \frac{\sin x}{x} = \frac{0}{0}$$

$$\stackrel{H}{=} \lim_{x \to 0} \frac{\cos x}{1} = \frac{1}{1} = 1$$

$g(x) = x \qquad g'(x) = 1/2\sqrt{x}$

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \overset{H}{=} \lim_{x\to\infty} \frac{1/x}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \frac{\infty}{\infty}$$

$$\frac{1}{x} \cdot \frac{2\sqrt{x}}{1}$$

$$f(x) = 2\sqrt{x} \qquad f'(x) = \frac{1}{\sqrt{x}}$$

$$g(x) = x \qquad g'(x) = 1$$

$$\overset{H}{=} \lim_{x\to\infty} 1/\sqrt{x} = 0$$

EXAMPLE 5 Find $\displaystyle\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x}$. $= \dfrac{0}{2} = 0$

-1

$$\lim_{x\to\pi^-} \frac{\cos x}{\sin x} = \frac{-1}{0} \to \infty \quad \times$$

$$\frac{\infty}{\infty},\ \frac{0}{0},\ \infty-\infty,\ 0^0,\ 0\cdot\infty,\ \infty^0,\ 1^\infty$$

Indeterminate Products (Type $0 \cdot \infty$)

This kind of limit is called an **indeterminate form of type $0 \cdot \infty$**. We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}$$

$x \to 0^+$

$$X = \frac{1}{1/x} \qquad x \ln x \to \frac{\ln x}{1/x}$$

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} \overset{H}{=} \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = -x \to 0$$

$f(x) = \ln x \quad f'(x) = 1/x$

$g(x) = 1/x \quad g'(x) = -1/x^2$

$$\lim_{x\to 0} \sin 5x \csc 3x$$

$\csc 3x = \dfrac{1}{\sin 3x}$

$$\lim_{x\to 0} \frac{\sin 5x}{\sin 3x} \overset{H}{=} \lim_{x\to 0} \frac{5\cos 5x}{3\cos 3x} = \frac{5}{3}$$

Indeterminate Differences (Type $\infty - \infty$)

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then the limit

$$\lim_{x \to a} [f(x) - g(x)]$$

$g(x) = (\ln x)(x-1) \qquad g'(x) = (1/x)(x-1) + (\ln x)(1)$

$$= \frac{(x-1) + x\ln x}{x}$$

$$\lim_{x\to 1^+} \frac{x-1}{x-1+x\ln x} \overset{H}{=} \lim_{x\to 1^+} \frac{1}{2+\ln x} = \boxed{\frac{1}{2}}$$

$f(x) = x-1 \qquad f'(x) = 1$

$g(x) = x-1+x\ln x \qquad g'(x) = 1 + \ln x + 1 = 2 + \ln x$

$1 - 0 + \ln x + \frac{1}{x}(x)$

$$\lim_{x\to 1^+} \frac{(x-1) - \ln x}{(\ln x)(x-1)}$$

$f(x) = x - 1 - \ln x$ $\qquad f'(x) = 1 - 1/x$

$g(x) = (\ln x)(x-1)$ $\qquad g'(x) = (1/x)(x-1) + (\ln x)(1)$

$$\frac{1 - \frac{1}{x}}{\frac{x-1}{x} + \ln x} = \frac{x-1}{x-1+x\ln x}$$

BitPaper
Share Knowledge

EXAMPLE 8 Calculate $\lim_{x\to\infty} (e^x - x)$.

$$e^x - x \quad \to x\left(\frac{e^x - x}{x}\right)$$

$$= x\left(\frac{e^x}{x} - 1\right)$$

$$\lim_{x\to\infty} x\left(\frac{e^x}{x} - 1\right)$$

$$\uparrow \ (\infty \ -1) = \infty(\infty) = \boxed{\infty}$$

$$\infty$$

Indeterminate Powers (Types 0^0, ∞^0, 1^∞)

Several indeterminate forms arise from the limit

$$\lim_{x \to a} [f(x)]^{g(x)}$$

1. $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ type 0^0
2. $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ type ∞^0
3. $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ type 1^∞

$x \to 0^+$ $\tan x$

$f(x) = \ln(1 + \sin 4x)$

$f'(x) = \dfrac{\dfrac{4\cos 4x}{1+\sin 4x}}{g'(x) = \sec^2 x} = \dfrac{\dfrac{4}{1}}{1}$

$g(x) = \tan x$

$e^{\lim\limits_{x \to 0} \ln y} = \boxed{e^4}$

$e^{\ln y} \to y$

Find $\lim_{x \to 0^+} x^x$.

$y = x^x$

$\ln y = \ln x^x$

$\ln y = \boxed{x \ln x}$

$\lim_{x \to 0^+} x \ln x \to 0$

$\ln y = 0$

$e^{\ln y} = e^0$

$y = \boxed{1}$

$$x^x = (e^{\ln x})^x = e^{x \ln x}$$

BitPaper
Share Knowledge