6 CHAPTER

Probability, Randomness, and Uncertainty

Random Experiment

A **random experiment** is defined as any activity or phenomenon that meets the following conditions.

1. There is one distinct outcome for each trial of the experiment.
2. The outcome of the experiment is uncertain.
3. The set of all distinct outcomes of the experiment can be specified and is called the **sample space**, denoted by *S*.

DEFINITION

Outcome

An **outcome** is any member of the sample space.

DEFINITION

Event

An **event** is a set of outcomes.

DEFINITION

Experiment 2: Toss a coin three times and observe the number of heads. Have we met the three conditions of a random experiment?

1. There will only be one outcome since, for example, it is not possible to have *exactly* one and *exactly* two heads on the same trial.
2. The outcome will be unknown *before* tossing the coin three times.
3. The sample space can be specified and is composed of eight outcomes:

   $S = \{\text{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH}\}$.

[redacted]

H H H
H H T
H T H
H T T

T H H
T H T
T T H
T T T

$$\frac{1}{8}$$

Relative Frequency

If an experiment is performed n times, under identical conditions, and the event A happens k times, the **relative frequency** of A is given by the following expression.

$$\text{Relative Frequency of } A = \frac{k}{n}$$

If the relative frequency converges as n increases, then the relative frequency is said to be the **probability** of A.

DEFINITION

experimental Probability

Classical Probability

Using the **classical approach** to probability, the probability of an event A, denoted $P(A)$, is given by

$$P(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes in the sample space}}.$$

DEFINITION

Theoretical Probability

Probability Law 1

A probability of zero means the event cannot happen. (For example, the probability of observing three heads in two tosses of a coin is zero.)

DEFINITION

Probability Law 2

A probability of one means the event must happen. (For example, if we toss a coin, the probability of getting either a head or tail is one.)

DEFINITION

Probability Law 3

All probabilities must be between zero and one inclusively. The closer the probability is to 1, the more likely the event. The closer the probability is to 0, the less likely the event. For an event A this is expressed as follows.

$$0 \le P(A) \le 1$$

DEFINITION

Probability Law 4

The sum of the probabilities of all outcomes must equal one. That is, if $P(A_i)$ is the probability of event A_i, and there are n such outcomes, then $P(A_1)+P(A_2)+\cdots+P(A_n)=1$.

DEFINITION

Compound event

A **compound event** is an event that is defined by combining two or more events.

DEFINITION

Let the event

$A = \{\text{annual income is greater than } \$50{,}000\}$

and

$B = \{\text{subscribes to more than one other sports magazine}\}$.

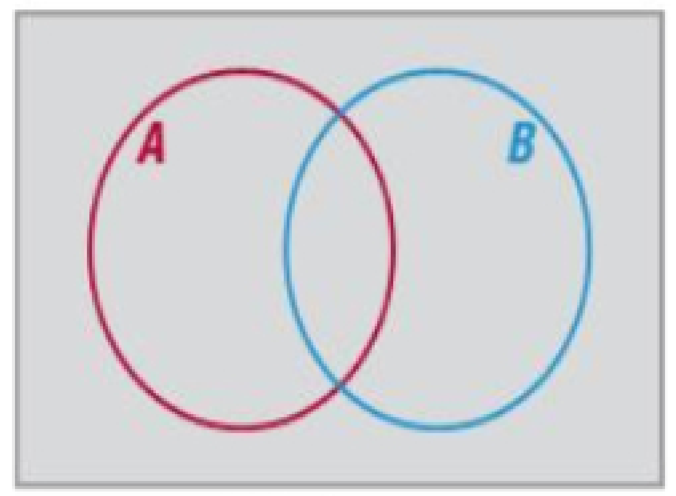


Figure 6.2.1

Union

The **union** of the events A and B is the set of outcomes that are included in event A or event B or both. Symbolically, the union of A and B is denoted $A \cup B$ and is read "A union B."

DEFINITION

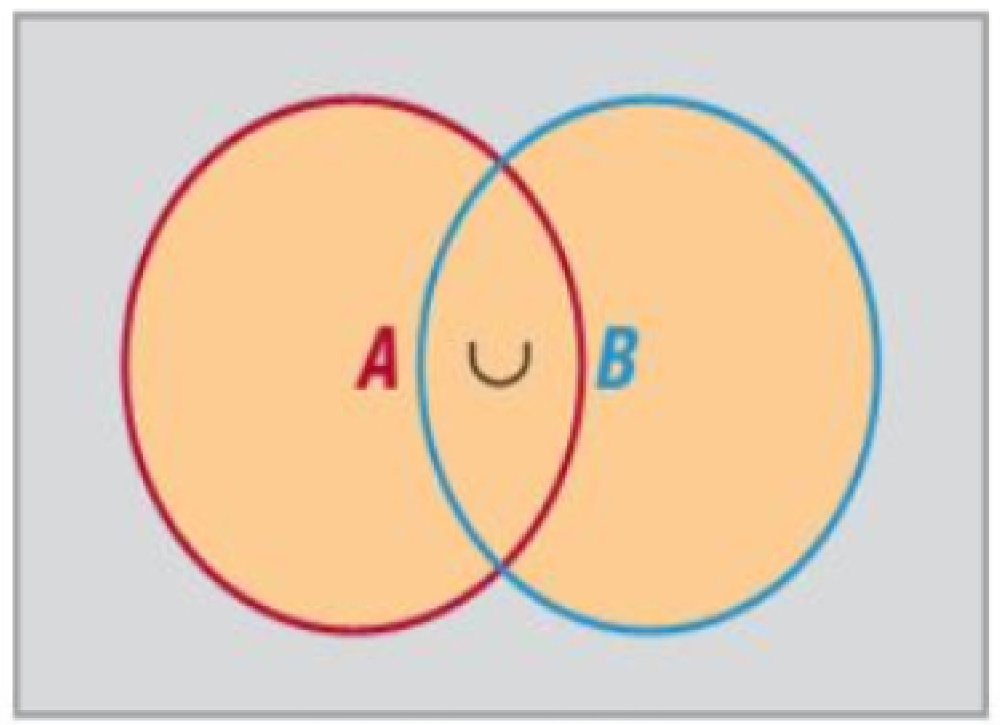


Figure 6.2.2

$A \boxed{\cup} B$

or

Intersection

The **intersection** of the events A and B is the set of all outcomes that are included in both A and B. Symbolically, the intersection of A and B is denoted $A \cap B$ and is read "A intersect B."

DEFINITION

Suppose the marketing director was interested in persons who possessed an annual income greater than $50,000 and subscribed to more than one other sports magazine. That set would be called the intersection of A and B.

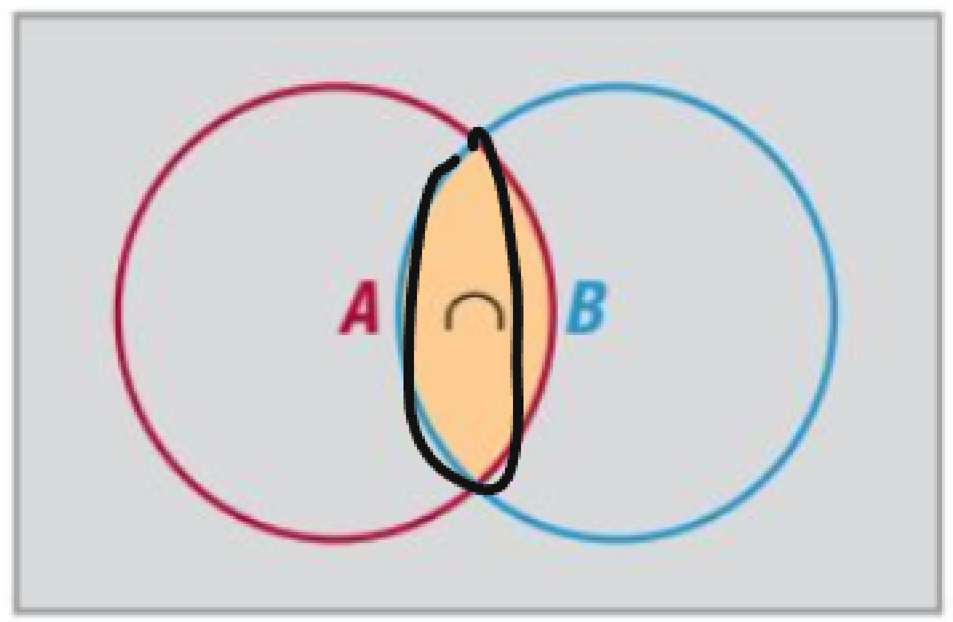


Figure 6.2.3

$A \cap B$

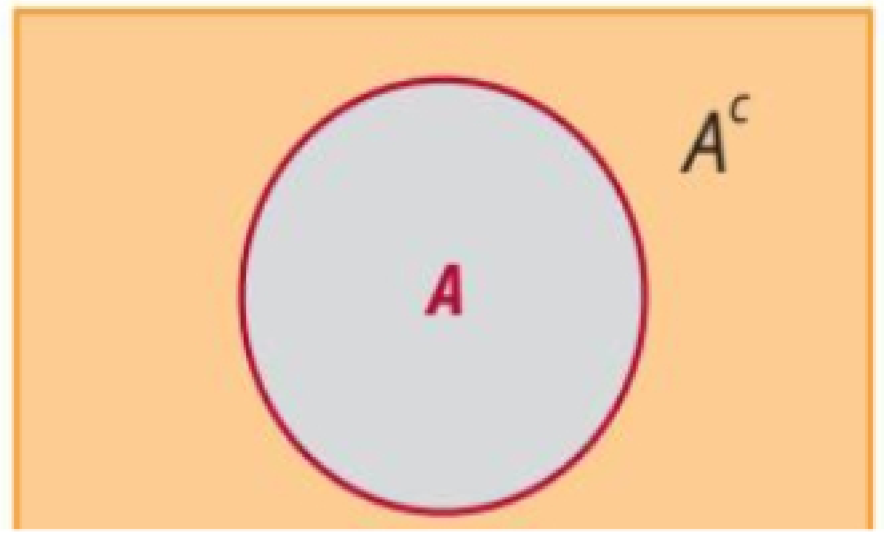
$\uparrow$

and

Complement

The **complement** of an event A is the set of all outcomes in the sample space that are not in A. Symbolically, the complement of the set A will be written as A^c.

DEFINITION



$A^c \rightarrow$ not A

Probability Law 5: Complement of an Event

The probability of A^c is given by $P(A^c) = 1 - P(A)$.

DEFINITION

Odds

The **odds in favor of** an event A occurring is given by $\dfrac{P(A)}{P(\text{not } A)} = \dfrac{P(A)}{P(A^c)}$.

The **odds against** an event A occurring is given by $\dfrac{P(\text{not } A)}{P(A)} = \dfrac{P(A^c)}{P(A)}$.

DEFINITION

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In the game of American roulette, the roulette wheel contains the numbers 1 through 36, alternating between red and black. There are two green spaces numbered 0 and 00.

- **a.** Calculate the probability of the roulette ball landing on a red pocket.
- **b.** Calculate the probability of the roulette ball not landing on a red pocket.
- **c.** Calculate the odds in favor of the roulette ball landing on red.

18 red   18 black   2 green

$P(\text{Red}) = \frac{18}{38}$

$P^c(\text{Red}) = \frac{20}{38}$

odd of red $\frac{18}{20}$

18 : 20

Mutually Exclusive

Two events are **mutually exclusive** if they have no outcomes in common.

DEFINITION

Mutual exclusivity is also called **disjointedness**. Figure 6.2.5 represents two **disjoint** events.

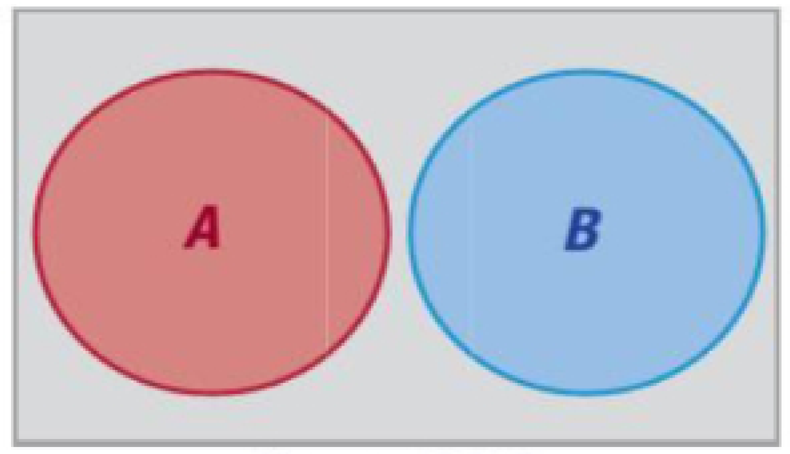


Figure 6.2.5

$A = 5$

$B = \text{Heads}$

$P(A \cup B) = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$

$= .167 + .5 = .667$

Probability Law 6: Union of Mutually Exclusive Events

If the events A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

DEFINITION

Probability Law 7: Intersection of Mutually Exclusive Events

If the events A and B are mutually exclusive, then

$$P(A \cap B) = 0.$$

DEFINITION

Probability Law 8: The General Addition Rule

For any two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

DEFINITION

$.52 + .63 - .40$

$P(A \cup B) = .75$

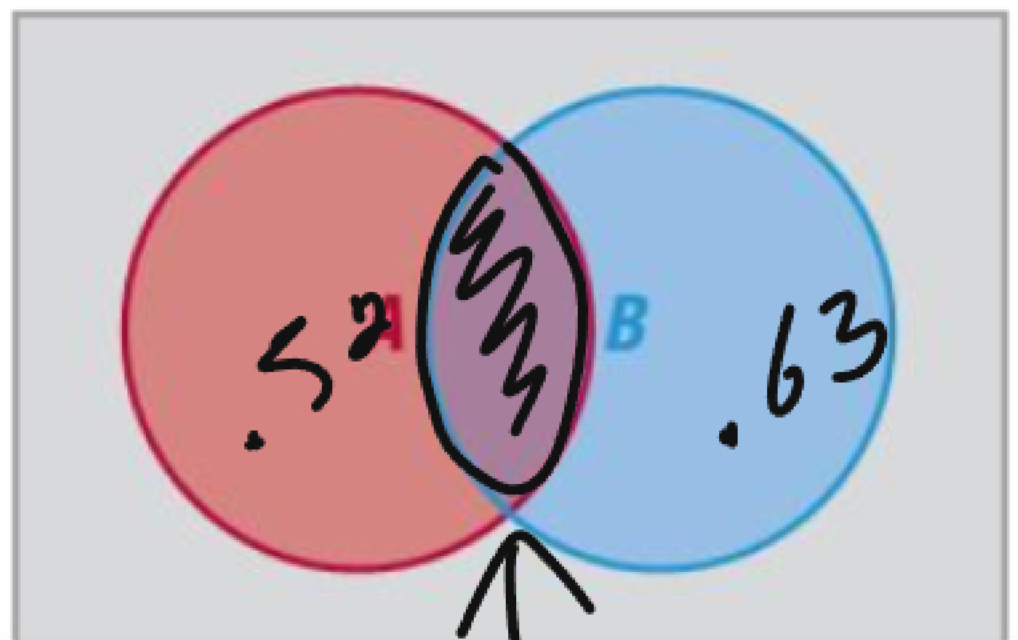


Figure 6.2.6

$P(A \cap B) = .40$

$P(A) = .52$

$P(B) = .63$

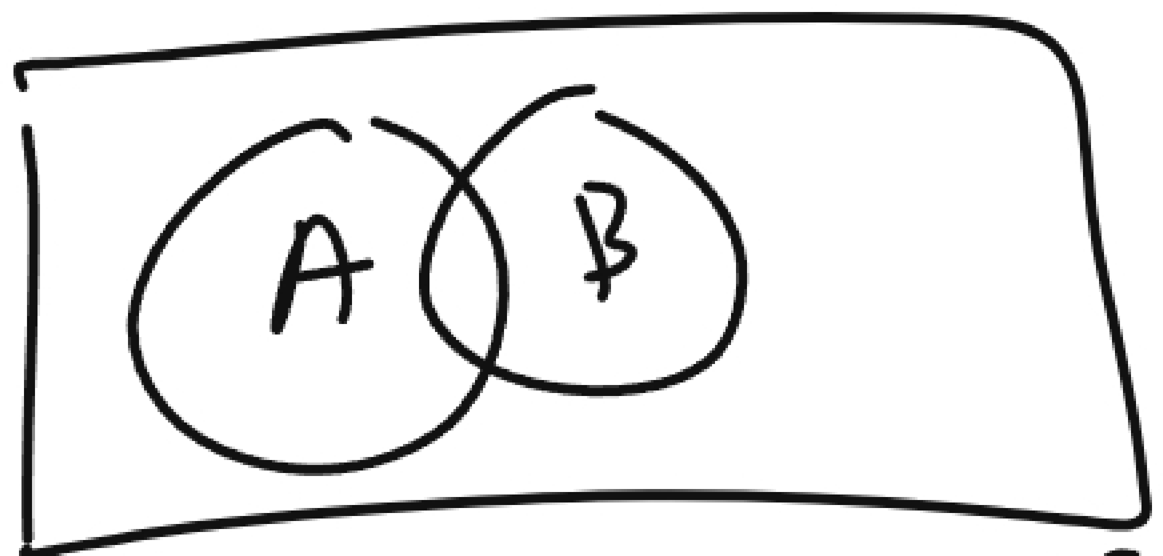
$P(A \text{ or } B) = 1.15$

20 sided Die

A = Even Numbers $\rightarrow P(A) = .5$

B = ≤ 7 $\rightarrow P(B) = .35$

$P(A \cap B) = .15$



$P(A \cup B) = .70$

A) 8, 10, 12, 14, 16, 18, 20

B) 1, 2, 3, 4, 5, 6, 7

15. A large life insurance company is interested in studying the insurance policies held by married couples. In particular, the insurance company is interested in the amount of insurance held by the husbands and the wives. The insurance company collects data for all of its 1000 policies where both the husband and the wife are insured. The results are summarized in the following table.

Life Insurance Coverage

		Amount of Life Insurance on Husband ($)			
		0–50,000	50,000–100,000	100,000–150,000	More than 150,000
Amount of Life Insurance on Wife ($)	0–50,000	400	200	50	50
	50,000–100,000	50	50	30	30
	100,000–150,000	20	10	25	25
	More than 150,000	20	10	15	15

c. For a randomly selected policy, what is the probability that the wife will have more than $150,000 of insurance or the husband will have more than $150,000 of insurance?

$P(\text{wife} > 150) = 60/1000$

$P(\text{Hus} > 150) = 120/1000$

$- 15/1000$

$P(A \cup B) = .165$

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Conditional Probability

The probability that an event will occur given that some other event has already occurred or is certain to occur, is a **conditional probability**.

Probability Law 9: Conditional Probability

The **conditional probability** of event A occurring, given that event B has already occurred is

$$\boxed{P(A|B)} = \frac{P(A \cap B)}{P(B)}$$

The notation $P(A|B)$ is read as *the probability of A given the occurrence of B*. The vertical bar within a probability statement will always mean *given*.

DEFINITION

Example 6.3.1

Suppose a marketing research firm has surveyed a panel of consumers to test a new product and produced the following **cross tabulation** indicating the number of panelists that liked the product, the number that did not like the product, and the number that were undecided.

Market Research Survey				
Age	**Like**	**Not Like**	**Undecided**	**Total**
18–34	213	197	103	513
35–50	193	184	67	444
Over 50	144	219	83	446
Total	**550**	**600**	**253**	**1403**

$P(A \cap B) = \frac{193}{1403}$

$P(B) = \frac{444}{1403}$

A → Like the product

B → 35 – 50

$P(A|B) = \frac{193}{444}$

$\frac{193}{1403} \div \frac{444}{1403} = \frac{193}{\cancel{1403}} \times \frac{\cancel{1403}}{444}$

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$P(A \cap B) = \frac{193}{1403}$

$P(B) = \frac{444}{1403}$

$A \rightarrow 35-50$
$B \rightarrow \text{Like}$

$P(A|B) = \frac{193}{550}$

Independent

Two events, A and B, are **independent** if and only if

$$P\left(A\middle|B\right) = P(A) \text{ and } P\left(B\middle|A\right) = P(B).$$

DEFINITION

In many cases, regarding the independence of two events, intuition and common sense will lead you to the correct determination. There are situations in which independence can only be discovered by formal application of the definition.

Probability Law 10: Multiplication Rule for Independent Events

If two events, A and B, are independent, then

$$P(A \cap B) = P(A) \cdot P(B).$$

If n events, $A_1, A_2, \ldots, A_n$, are independent, then

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1) \cdot P(A_2) \cdot \ldots \cdot P(A_n).$$

DEFINITION

Example 6.3.5

A coin is flipped, a die is rolled, and a card is drawn from a standard deck of 52 cards. Find the probability of getting a tail on the coin, a five on the die, and a jack of clubs from the deck of cards.

$P(A) = .50$ $P(B) = .167$ $P(C) = .019$

$P(A \cap B \cap C) = (.50)(.167)(.019)$

$= .0016$

Example 6.3.7

This is an actual case that stirred up quite a controversy.

People vs. Collins (1968)

On June 18, 1964, at about 11:30 AM, Mrs. Juanita Brooks was assaulted and robbed while walking through an alley in the San Pedro area of Los Angeles. Mrs. Brooks described her assailant as a young woman with a blond pony tail. At about the same time, John Bass was watering his lawn and witnessed the assault. He described the assailant as a Caucasian woman with dark-blond hair. As she ran from the alley she jumped into a yellow automobile driven by a black man with a mustache and a beard.

Several days later the police arrested two individuals based on the descriptions provided by the assailant and the witness. The two suspects were eventually charged with the crime. During the trial the prosecution called a professor of mathematics to testify. The prosecutor set forth the following probabilities for the characteristics of the assailants.

Assailant Characteristics Data	
Characteristic	**Probability**
Yellow automobile	0.10
Man with mustache	0.25
Girl with ponytail	0.10
Girl with blonde hair	0.33
Black man with beard	0.10
Interracial couple in a car	0.001

How did the prosecution use these probabilities to argue its case?

.000 000 0 0825

Example 6.3.8

In a production process, a product is assembled by using four different independent parts (A, B, C, and D). In order for the product to operate properly, each part must be free of defects. The probabilities of the parts being nondefective are given by $P(A) = 0.9$, $P(B) = 0.7$, $P(C) = 0.8$, and $P(D) = 0.9$.

a. What is the probability that all four parts are defective?
b. What is the probability that the product does not work?

$$\begin{aligned} P(\text{all four parts have defects}) &= P(A^c)\cdot P(B^c)\cdot P(C^c)\cdot P(D^c) \\ &= (1-0.9)(1-0.7)(1-0.8)(1-0.9) = 0.0006 \end{aligned}$$

Thus, the probability that all four parts have defects is 0.0006, or 0.06%.

$$\begin{aligned} P(\text{product does not work}) &= P(\text{at least one part does not work}) \\ &= 1 - P(\text{all parts work}) \\ &= 1 - P(A \cap B \cap C \cap D) \\ &= 1 - (0.9)(0.7)(0.8)(0.9) = 1 - 0.4536 = 0.5464 \end{aligned}$$

Thus, there is nearly a 55% chance that the product will not work.

Probability Law 11: Multiplication Rule for Dependent Events

If two events, A and B, are **dependent**, then

$$P(A \cap B) = P(A) \cdot P(B \mid A) = P(B) \cdot P(A \mid B)$$

DEFINITION

Example 6.3.10

Assume that there are 17 men and 24 women in the Lions Club. Two members are chosen at random each year to serve on the hospitality committee. What is the probability of choosing two members at random and the first being a man and the second being a woman?

$$\frac{17}{41} \times \frac{24}{40} =$$

The Fundamental Counting Principle

E_1 is an event with n_1 possible outcomes and E_2 is an event with n_2 possible outcomes. The number of ways the events can occur in sequence is $n_1 \cdot n_2$. This principle can be applied for any number of events occurring in sequence.

PROCEDURE

Select five numbers between 1 and 69 for the white balls, then select one number between 1 and 26 for the red Powerball.

$$\boxed{69} \times \boxed{68} \times \boxed{67} \times \boxed{66} \times \boxed{65} \times \boxed{26}$$

35,064,160,560

2 Letters 5 Numbers

$\boxed{26}\ \boxed{26}\quad \boxed{10}\ \boxed{10}\ \boxed{10}\ \boxed{10}\ \boxed{10}$

67,600,000

Permutation

A **permutation** is a specific order or arrangement of objects in a set. There are $n!$ permutations of n unique objects.

DEFINITION

$_nP_r$

n = number of options

r = number of choices you want

$_{10}P_3$ or nPr(10, 3)

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Combinations

A **combination** is a collection or grouping of objects where the order is ***not*** important.

DEFINITION

${}_{n}C_{r}$

n = number of options

r = number of choices you want

${}_{10}C_{3}$ or $nCr(10, 3)$

Example 6.4.6

Seven sprinters have advanced to the final heat at a track meet. How many ways can they finish in first, second, and third place?

$${}_7P_3 \quad \text{or} \quad {}_nP_r(7,3)$$

$$\boxed{210}$$

Distinguishable Permutations

If given n objects, with n_1 alike, n_2 alike, … , n_k alike, then the number of **distinguishable permutations** of all n objects is $\dfrac{n!}{(n_1!n_2!n_3!\ldots n_k!)}$ FORMULA

Example 6.4.7

How many distinguishable permutations can be made from the word *Mississippi*?

Solution

There are 11 letters in the word *Mississippi*, one M, four I's, four S's, and two P's. So there are,

$$\frac{11!}{(1!4!4!2!)} = 34,650$$

distinguishable permutations of the letters in *Mississippi*.

Example 6.4.8

In South Carolina's *Palmetto Cash 5* lottery, a player selects five different numbers from 1 to 38 (inclusive). If the numbers selected match the player's numbers in any order, the player wins.

a. What is the total number of winning combinations?
b. What is the probability of winning?

$_{38}C_5 =$

$\mathrm{nCr}(38,5) = 501\,942$

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23. The engineering club at a local high school must choose 2 representatives from each of the sophomore, junior, and senior classes to attend a national convention. If there are 6 sophomores, 5 juniors, and 7 seniors in the club, in how many ways can the group be chosen for the convention?

$_6C_2 = 15$

$_5C_2 = 10$

$_7C_2 = 21$

$15 \times 10 \times 21 = 3150$