Quantitative Literacy:
Thinking Between the Lines

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Chapter 3:
Linear and Exponential Change:

Comparing Growth Rates

Chapter 3: Linear and Exponential Changes
Lesson Plan

- Lines and linear growth: What does a constant rate mean?
- Exponential growth and decay: Constant percentage rates
- Logarithmic phenomena: Compressed scales

Learning Objectives:

- Understand linear functions and consequences of a constant growth rate.
- Interpret linear functions.
- Calculate and interpret the slope.
- Understand linear data and trend lines for linear approximations.

3.1 Lines and linear growth: What does a constant rate mean?

- A **linear function** is a function with a constant growth rate.
- A **graph** of a linear function is a straight line.
- **Example (Determining linear or not):** Find the growth rate of the function. Make a graph of the function. Is the function linear?

  For my daughter's wedding reception, I pay $500 rent for the building plus $15 for each guest. This describes the total cost of the reception as a function of the number of guests.

- **Solution:** The growth rate is the extra cost incurred for each additional guest, that is $15. So, the growth rate is **constant**.

  The additional cost means each additional guest.

  The total cost of the reception is a **linear** function of the number of guests.

3.1 Lines and linear growth: What does a constant rate mean?

- **Example (Determining linear or not):** Find the growth rate of the function and give its practical meaning. Make a graph of the function. Is the function linear?

  My salary is initially $30,000, and I get a 10% salary raise each year for several years. This describes my salary as a function of time.

- **Solution:** The growth rate:

$$1^{\text{st}} \text{ year increased} = 10\% \text{ of } \$30{,}000 = \$3{,}000$$

$$\therefore \quad 1^{\text{st}} \text{ year salary} = \$33{,}000$$

$$2^{\text{nd}} \text{ year increased} = 10\% \text{ of } \$33{,}000 = \$3{,}300$$

$$\therefore \quad 2^{\text{nd}} \text{ year salary} = \$36{,}300$$

The growth rate is **not the same each year**. So, the graph is **not** a straight line. Thus, my salary is **not** a linear function of time in years.

3.1 Lines and linear growth: What does a constant rate mean?

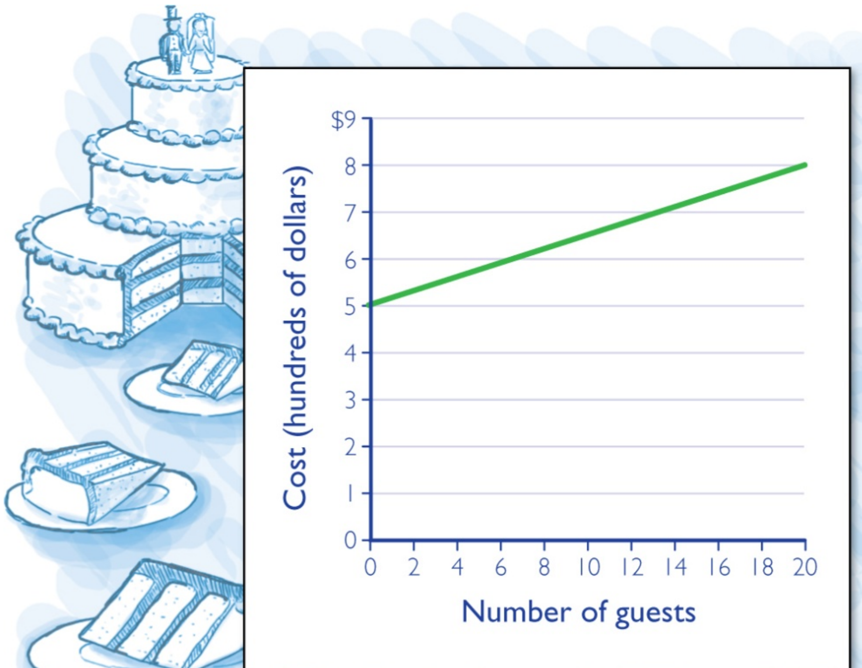

FIGURE 3.5 Cost is a linear function of number of wedding guests.

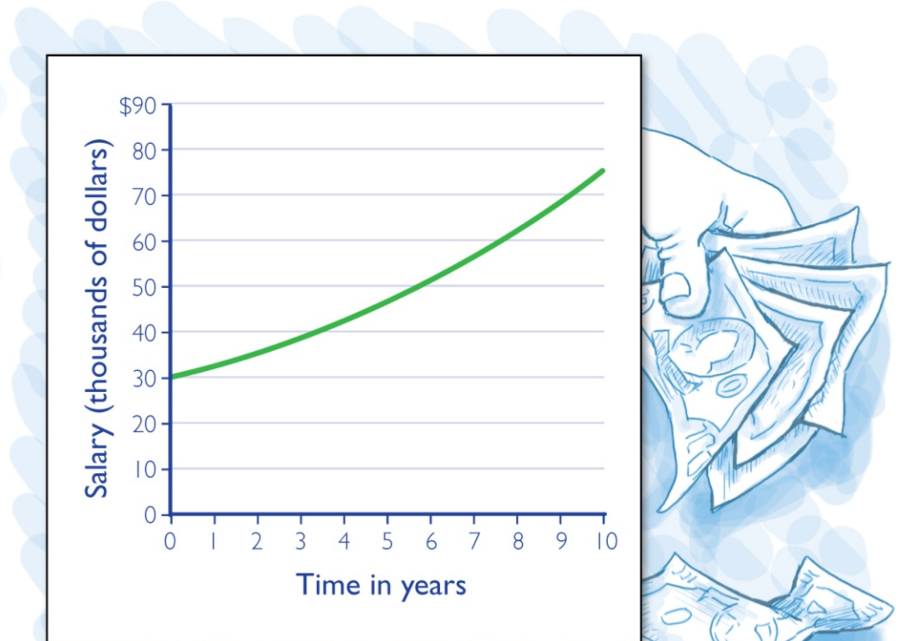

FIGURE 3.6 Salary is not a linear function of time.

3.1 Lines and linear growth: What does a constant rate mean?

Formula for Linear Function

$$y = \text{Growth rate } \times x + \text{Initial value}$$

If *m* is the growth rate or slope and *b* is the initial value,

$$y = mx + b.$$

- **Example:** Let *L* denote the length in meters of the winning long jump in the early years of the modern Olympic Games. Suppose *L* is a function of the number *n* of Olympic Games since 1990, an approximate linear formula is $L = 0.14n + 7.20$.

  Identify the initial values and growth rate, and explain in practical terms their meaning.

- **Solution:** The initial value is 7.20 meters. The growth rate is 0.14 meter per Olympic Game. It means that the length of the winning long jump increased by 0.14 meters from one game to the next.

3.1 Lines and linear growth: What does a constant rate mean?

- **Example:** A rocket starting from an orbit 30,000 kilometers (km) above the surface of Earth blasts off and flies at a constant speed of 1000 km per hour away from Earth.

1. Explain why the function giving the rocket's distance from Earth in terms of time is linear.
2. Identify the initial value and growth rate.
3. Find a linear formula for the distance.

The Saturn V carried the first men to the moon in 1969.

- **Solution:**

1. We first choose letters to represent the function and variable. Let d be the distance in km from Earth after t hours.

   $\text{The growth rate} = \text{velocity} = 1000\ \text{km/hour} = \text{a constant}$

   Thus, d is a linear function of t.

2. $\text{The Initial value} = 30{,}000\ \text{km}$

   $= \text{the height above Earth at blastoff}$

3. $d = \text{Growth rate} \times t + \text{Initial value}$

   $= 1000\,t + 30{,}000$

Chapter 3 Linear and Exponential Changes

3.1 Lines and linear growth: What does a constant rate mean?

- **Interpreting and using the Slope**

- The slope of a linear function is using:

$$\text{Slope} = \text{Growth rate} = \frac{\text{Change in function}}{\text{Change in independent variable}}$$

- Write the equation of a linear function as: $y = mx + b$, the formula for the slope becomes

$$m = \text{Slope} = \frac{\text{Change in } y}{\text{Change in } x}$$

- Each 1-unit increase in *x* corresponds to a change of *m* units in *y*.

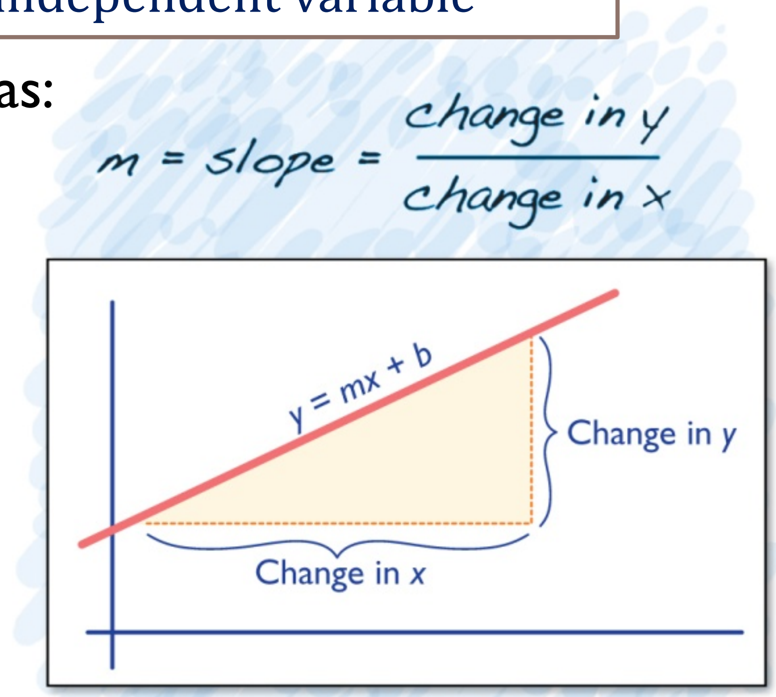

FIGURE 3.9 How slope is calculated.

3.1 Lines and linear growth: What does a constant rate mean?

- **Example:** Suppose your car's 20-gallon tank is full when you begin a road trip.

  Assume that you are using gas at a constant rate, so the amount of gas in your tank is a linear function of the time in hours you have been driving.

  After traveling for two hours, your fuel gauge reads three-quarters full.

  Find the slope of the linear function and explain in practical terms what it means.

- **Solution:**
- When the tank is ¾ full, there are 15 gallons of gas left in the tank.
- The amount of gas in the tank has decreased by 5 gallons.
- The change in gas is –5 gallons over 2 hours driving.

$$m = \frac{\text{Change in gas}}{\text{Change in time}} = \frac{-5 \text{ gallons}}{2 \text{ hours}} = -2.5 \text{ gallons per hour}$$

- This means we are using 2.5 gallons of gas each hour.

3.1 Lines and linear growth: What does a constant rate mean?

- **Example:** Measuring temperature using the Fahrenheit scale is common in the United States, but use of the Celsius scale is more common in most other countries.

  The temperature in degrees Fahrenheit (°F) is a linear function of the temperature in degrees Celsius (°C).

  On the Celsius scale, 0°C is the freezing temperature of water. This occurs at 32°F on the Fahrenheit scale. Also, 100°C is the boiling point of water. This occurs at 212°F.

3.1 Lines and linear growth: What does a constant rate mean?

- **Example (cont.):**

1. What is the slope of the linear function giving the temperature in degrees Fahrenheit in terms of the temperature in degrees Celsius?
2. Choose variable and function names, and find a linear formula that converts degrees Celsius to degrees Fahrenheit. Make a graph of the linear function.
3. A news report released by Reuters on March 19, 2002, said that the Antarctic peninsula had warmed by 36°F over the past half-century. The British writer saw a report that the temperature had increased by 2.2°C. Verify that a temperature of 2.2°C is about 36°F.
4. What increase in Fahrenheit temperature should the writer have reported in part 3?

- **Solution:**

1. To find a slope:

An increase on the Celsius scale from 0 to 100 degrees corresponds to an increase on the Fahrenheit scale from 32 to 212 degrees.

$$\text{slope} = \frac{\text{Change in degrees Fahrenheit}}{\text{Change in degrees Celsius}} = \frac{180^{\text{o}}\text{F}}{100^{\text{o}}\text{C}} = 1.8^{\text{o}}\text{F}/^{\text{o}}\text{C}$$

Thus, a 1-degree increase on the Celsius scale corresponds to a 1.8-degree increase on the Fahrenheit scale.

- **Solution:**

2. Use F for the temperature in degrees Fahrenheit and C for the temperature in degrees Celsius.

   The linear relation we expressed by the formula:

$$F = \text{Slope} \times C + \text{Initial value}$$

$$F = 1.8C + 32$$

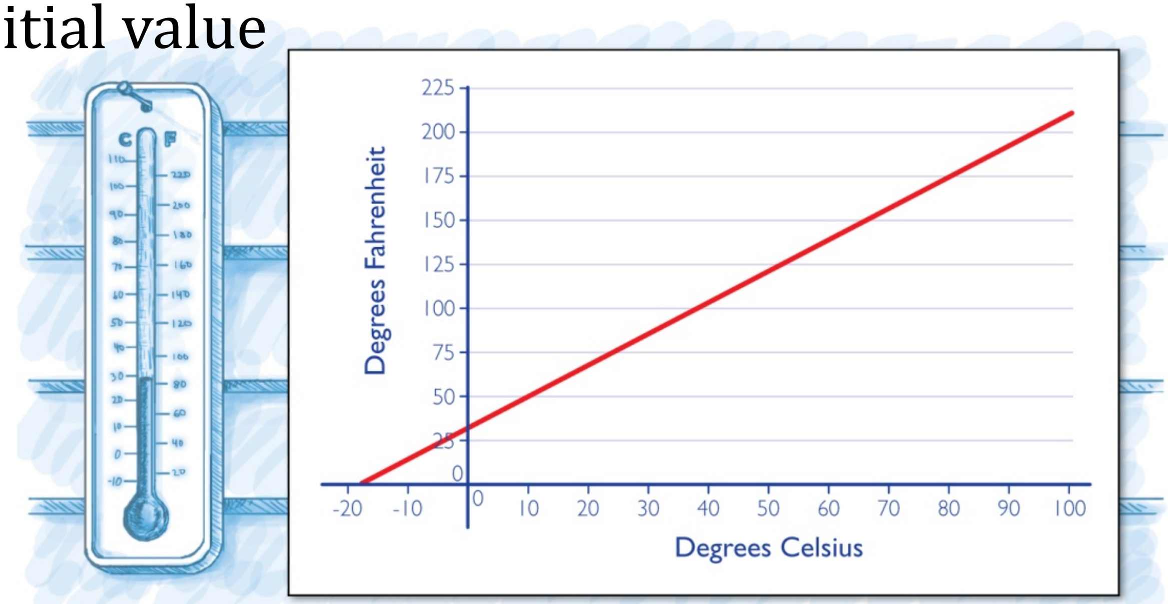

FIGURE 3.11 Fahrenheit temperature is a linear function of Celsius temperature.

- **Solution:**

3. Verify that a temperature of 2.2 degrees Celsius is about 36 degrees Fahrenheit.

   We put 2.2 degrees Celsius into the formula

$$\mathrm{F} = \mathrm{Slope} \times \mathrm{C} + \mathrm{Initial\ value}$$

   to convert Celsius to Fahrenheit:

$$\begin{aligned} \mathrm{F} &= 1.8\,\mathrm{C} + 32 \\ &= (1.8 \times 2.2) + 32 \\ &= 35.96^{o}\mathrm{F} \end{aligned}$$

- **Solution:**

4. What increase in Fahrenheit temperature should the writer have reported in part 3?

The slope of the linear function is $1.8^{\circ}\mathrm{F}$ per $^{\circ}\mathrm{C}$.

An increase of $2.2^{\circ}\mathrm{C}$ corresponds to an increase of

$$2.2 \times 1.8 = 3.96^{\circ}\mathrm{F}.$$

The writer should have reported a warming of about $4^{\circ}\mathrm{F}$.

3.1 Lines and linear growth: What does a constant rate mean?

- Given a set of data points, the **regression line** (or **trend line**) is a line that comes as close as possible to fitting those data.
- **Example:** The following table shows the running speed of various animals vs. their length. Show the scatterplot and find the formula for the trend line. Explain in practical terms the meaning of the slope.

Animal	Length (inches)	Speed (feet per second)
Deer mouse	3.5	8.2
Chipmunk	6.3	15.7
Desert crested lizard	9.4	24.0
Grey squirrel	9.8	24.9
Red fox	24.0	65.6
Cheetah	47.0	95.1

- **Solution:**

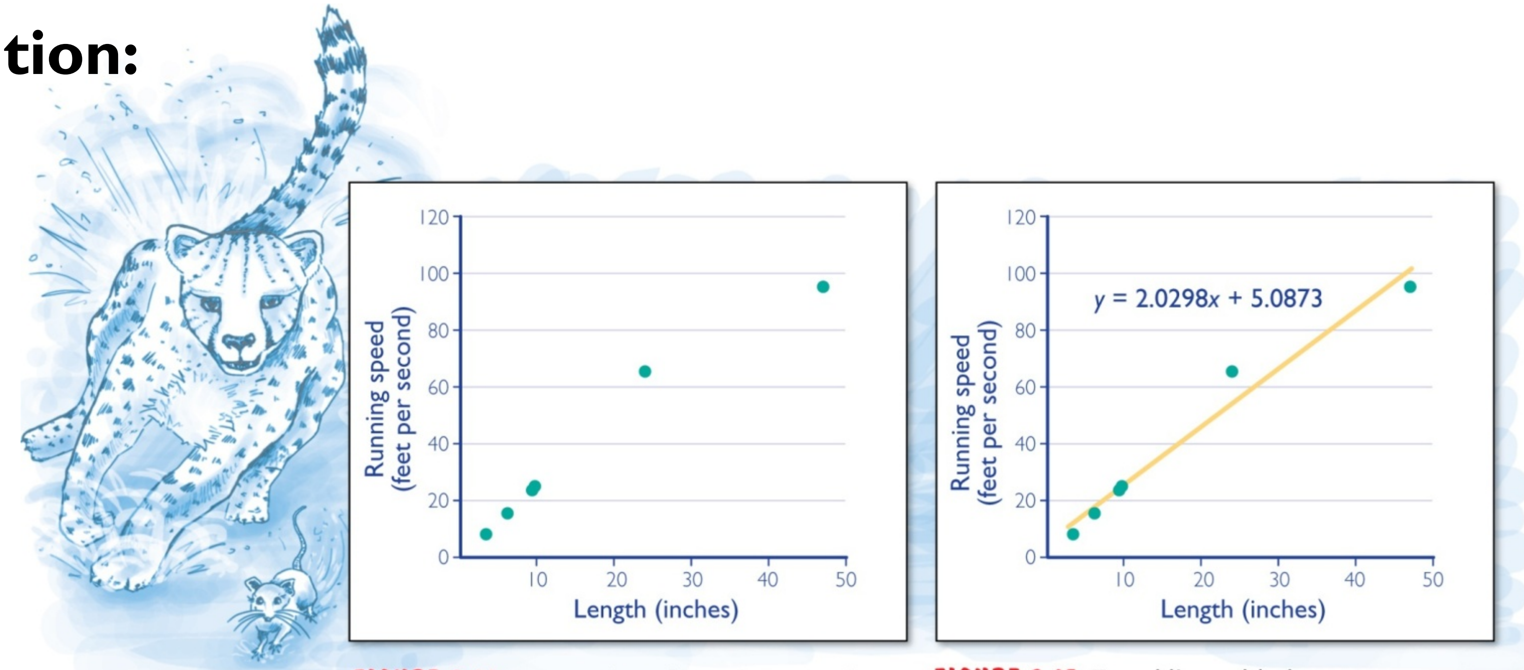

FIGURE 3.14 Scatterplot of running speed versus length.

FIGURE 3.15 Trend line added.

The points do not fall on a straight line, so the data in the table are not exactly linear. In Figure 3.15, we have added the trend line produced by the spreadsheet program Excel.

- **Solution:** The equation of the trend line is:

$$y = 2.03x + 5.09$$

This means that running speed S in feet per second can be closely estimated by:

$$S = 2.03\ L + 5.09,$$

where L is the length measured in inches.

The slope of the trend line is 2.03 feet per second per inch.

This value for the slope means that an animal that is 1 inch longer than another would be expected to run about 2.03 feet per second faster.

Learning Objectives:

- Understand exponential functions and consequences of constant percentage change.
- Calculate exponential growth, exponential decay, and the half-life.

3.2 Exponential growth and decay: Constant percentage rates

- An **exponential function** is a function that changes at a constant percentage rate.

- **Example:** If a population triples each hour, does this represent constant percentage growth?

If so, what is the percentage increase each hour?

Is the population size an exponential function of time?

- **Solution:** The population changes each hour:

$$\text{Population next hour} = 3 \times \text{Current population}$$

Suppose we start with 100 individuals:

$$\text{Initial population} = 100$$

$$\text{Population after 1 hour} = 3 \times 100 = 300$$

$$\text{Population after 2 hours} = 3 \times 300 = 900$$

Let's look at this in terms of growth:

Growth over 1st hour $= 300 - 100 = 200 = 200\%$ increase over 100

Growth over 2nd hour $= 900 - 300 = 600 = 200\%$ increase over 300

The population is growing at a constant percentage rate, 200% each hour. Thus, the population is an exponential function of time.

Exponential Formulas

- The formula for an exponential function *y* of *t* is:

$$\mathrm{y} = \text{Initial value} \times \text{Base}^{t}$$

- An exponential function *y* of *t* is characterized by the following property: When *t* increases by 1, to find the new value of *y*, we multiply the current value by the base.

$$y - \text{value for } t + 1 = \text{Base} \times \mathrm{y} - \text{value for } t$$

- **Example:** The value of a certain investment grows according to the rule:

$$\text{Next year's balance} = 1.07 \times \text{Current balance}$$

1. Find the percentage increase each year, and explain why the balance is an exponential function of time.
2. Assume that the original investment is \$800. Find an exponential formula that gives the balance in terms of time.
3. What is the balance after 10 years?

- **Solution:**

1. The next year's balance $=1.07\times$ this year's balance.

The next year's balance is 107% of this year's balance.

That is an increase of 7% per year. Because the balance grows by the same percentage each year, it is an exponential function of time.

2. Let $B=$ the balance in dollars after t years.

$$B=\text{Initial value }\times\text{Base}^t$$

The initial value $=\$800$. The base $=1.07$. This gives the formula:

$$B=800\times1.07^t$$

3. Balance after 10 years $=800\times1.07^{10}=\$1573.72$.

- **Example:** Consider the investment from the previous example where the balance B after t years is given by:

$$B = 800 \times 1.07^{t} \text{ dollars}$$

What is the growth of the balance over the first 10 years? Compare this with the growth from year 40 to year 50.

- **Solution:** The balance after 10 years was **$1573.72.**

$$\text{Growth over first 10 years} = \$1573.72 - 800 = \$773.72.$$

To calculate the growth from year 40 to year 50:

$$\text{Balance after 40 years} = 800 \times 1.07^{40} = \$11{,}979.57$$
$$\text{Balance after 50 years} = 800 \times 1.07^{50} = \$23{,}565.62$$

That is an increase of $\$23{,}565.62 - \$11{,}979.57 = \$11{,}586.05$.
That is almost 15 times the growth over the first 10 years.

Exponential Growth

1. A quantity grows exponentially when it increases by a constant percentage over a given period.
2. If r is the percentage growth per period then the base of the exponential function is $1 + \mathrm{r}$. For exponential growth, the base is always greater than 1.

$$\text{Amount} = \text{Initial value} \times (1 + r)^{t}$$

Here, t is the number of periods.

3. Typically, exponential growth starts slowly and then increases rapidly.

- **Example:** U.S. health-care expenditures in 2010 reached 2.47 trillion dollars. In the near term this is expected to grow by 6.5% each year. Assuming that this growth rate continues, find a formula that gives health-care expenditures as a function of time. If this trend continues, what will health-care expenditures be in 2030?

- **Solution:** Let *H* be the expenditures *t* years after 2010.

$$H = \text{Initial value } \times (1 + r)^t = 2.47 \times (1 + 0.065)^t.$$

To predict health-care expenditures in 2030, use $t = 20$ in the formula for *H*:

$$\text{Expenditures in } 2030 = 2.47 \times (1 + 0.065)^{20} \text{ trillion dollars.}$$

- **Solution:**

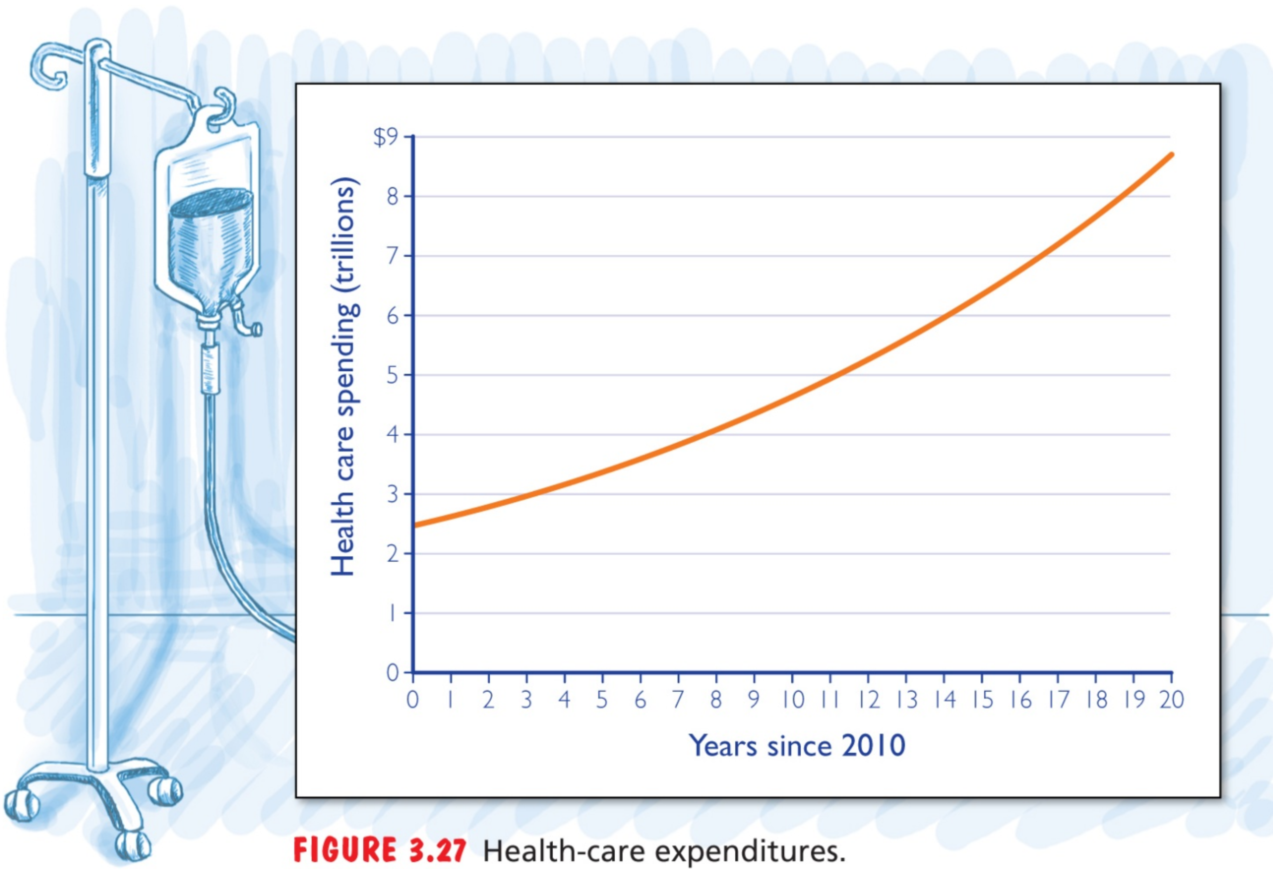

FIGURE 3.27 Health-care expenditures.

Exponential Decay

1. A quantity decays exponentially when it decreases by a constant percentage over a given period.
2. If r is the percentage decay per period, then the base of the exponential function is $1-r$. For exponential decay, the base is always less than 1.

$$\text{Amount} = \text{Initial value} \times (1-r)^t$$

   Here, t is the number of periods.
3. Typically, exponential decay is rapid first but eventually slows.

- **Example:** After antibiotics are administered, the concentration in the bloodstream declines over time. Suppose that 70 milligrams (mg) of amoxicillin are injected and that the amount of the drug in the bloodstream declines by 49% each hour.

  Find an exponential formula that gives the amount of amoxicillin in the bloodstream as a function of time since the injection. Another injection will be required when the level declines to 10 mg. Will another injection be required before five hours?

- **Solution:** Let A be the amount of amoxicillin in the bloodstream after t hours. The base of the exponential function is:

$$1 - r = 1 - 0.49 = 0.51$$

The initial value = 70 mg:

$$A = \text{Initial value } \times (1 - r)^t = 70 \times 0.51^t$$

To find the amount of amoxicillin after 5 hours:

$$A = 70 \times 0.51^5 = 2.4 \text{ mg}$$

The result is about 2.4 mg, which is less than the minimum of 10 mg. Thus, another injection will be needed before 5 hours.

- **Solution:**

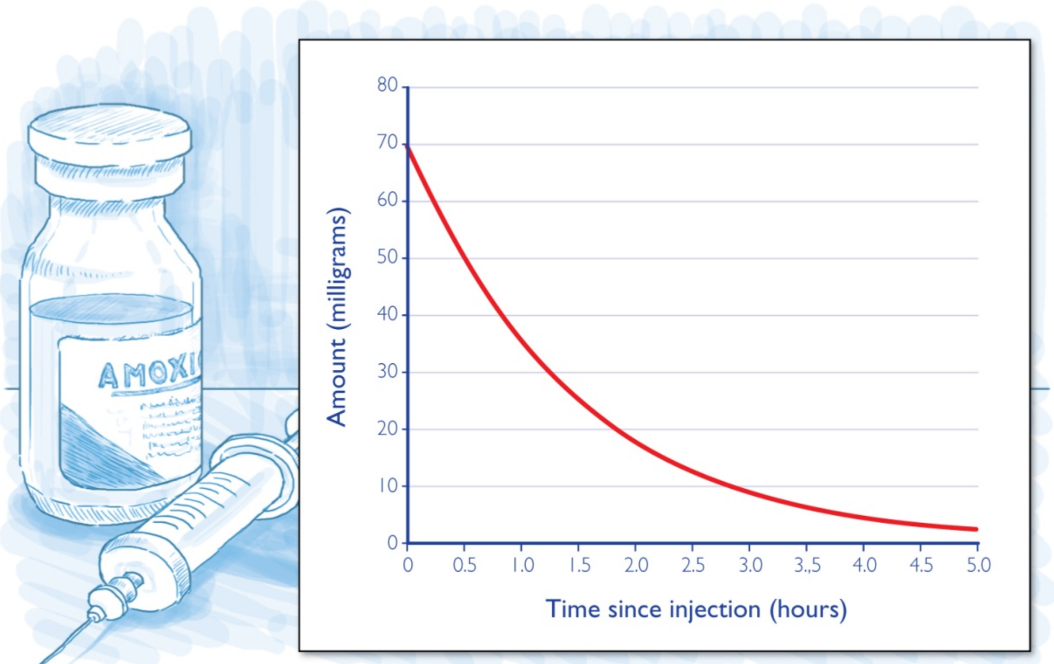

FIGURE 3.29 Amoxicillin in the blood.

The graph shows that the level declines to 10 mg in about 3 hours.

- The **half-life** of a radioactive substance is the time it takes for half of the substance to decay.
- After *h* half-lives, the amount of a radioactive substance remaining is given by the exponential formula:

Half-life formula

$$\text{Amount remaining} = \text{Initial amount } \times \left(\frac{1}{2}\right)^{t}$$

- We can find the amount remaining after *t* years by first expressing *t* in terms of half-lives and then using the formula above.

- **Example:** The carbon-14 in the organism decays, with a half-life of 5770 years. Suppose a tree contains C_0 grams of carbon-14 when it was cut down. What percentage of the original amount of carbon-14 would we find if it was cut down 30,000 years ago?
- **Solution:**

$$\text{The amount } C \text{ remaining after } h \text{ half} - \text{life} = C_o \times \left(\frac{1}{2}\right)^h$$

$$5770 = 1 \text{ half-life } \Longrightarrow \ 30{,}000 \text{ years}/5770 = 5.20 \text{ half-lives}$$

$$\text{Amount after } 30{,}000 \text{ years} = C_o \times \left(\frac{1}{2}\right)^{5.20} = 0.027C_o \text{ grams.}$$

Thus, about 2.7% of the original amount of carbon-14 remains after 30,000 years.

Learning Objectives:

- Understand the use of logarithms in compressed scales.
- Understand the Richter scale and calculate its magnitude in terms of relative intensity.
- Understand and calculate decibel reading in terms of relative intensity.
- Solve exponential equations.
- Calculate the doubling time.

- The **common logarithm** of a positive number *x*, written $\log x$, is the exponent of 10 that gives *x*:

$$\log x = t \quad \text{if and only if} \quad 10^t = x$$

- **Example:** Calculating logarithms

1. $\log 10 = 1$ because $10^1 = 10$.
2. $\log 100 = 2$ because $10^2 = 100$.
3. $\log 1000 = 3$ because $10^3 = 10000$.
4. $\log \frac{1}{10} = -1$ because $10^{-1} = \frac{1}{10}$.

3.3 Logarithmic phenomena: Compressed scales

- The **relative intensity** of an earthquake is a measurement of ground movement.
- The **magnitude** of an earthquake is the logarithm of relative intensity:

$$\text{Magnitude} = \log(\text{Relative intensity}),$$

$$\text{Relative intensity} = 10^{\text{Magnitude}}$$

- **Example:** If an earthquake has a relative intensity of 6700, what is its magnitude?
- **Solution:**

$$\text{Magnitude} = \log(\text{Relative intensity}) = \log(6700) \approx 3.8$$

- **Meaning of magnitude changes**

1. An increase of 1 unit on the Richter scale corresponds to increasing the relative intensity by a factor of 10.
2. An increase of t units in magnitude corresponds to increasing the relative intensity by a factor of 10^t.

- **Example:** In 1994 an earthquake measuring 6.7 on the Richter scale occurred in Northridge, CA. In 1958 an earthquake measuring 8.7 occurred in the Kuril Islands.

  How did the intensity of the Northridge quake compare with that of the Kuril Islands quake?

- **Solution:** The Kuril Islands quake was $8.7 - 6.7 = 2$ points higher. Increasing magnitude by 2 points means that relative intensity increases by 10^2. The Kuril Islands quake was 100 times as intense as the Northridge quake.

- The **decibel** rating of a sound is 10 times the logarithm of its relative intensity:

$$\text{Decibels} = \log(\text{Relative intensity}),$$

$$\text{Relative intensity} = 10^{0.1\times\text{Decibels}}$$

- **Increasing Decibels**

1. An increase of 1 decibel multiplies relative intensity by 1.26.
2. An increase of t decibels multiplies relative intensity by 1.26^t.

- **Example:** An idling bulldozer produces a sound that is about 85 decibels. As shown in the table below, how does the relative intensity of an idling bulldozer compare with the vacuum cleaner?

Sound	Decibels	Relative intensity
Threshold of audibility	0	1
Whisper	20	100
Vacuum cleaner	80	10^8
Jet takeoff	140	10^{14}

- **Solution:** Increasing the decibels by $85 - 80 = 5$ multiplies the intensity by 1.26^5 or about 3.2.

The sound of the bulldozer is about 3.2 times as intense as that of the vacuum cleaner.

- **Example:** Suppose we have a stereo speaker playing music at 60 decibels. What decibel reading would be expected if we add a second speaker?

- **Solution:** $\text{Relative intensity} = 1.26^{\text{Decibels}} = 1.26^{60}$

  With second speaker added, the new relative intensity is doubled:

$$\text{New relative intensity} = 2 \times 1.26^{60}$$

Thus, the decibel reading of the pair of speakers:

$$\text{Decibels} = 10\log(\text{Relative intensity})$$
$$= 10\log(2 \times 1.26^{60})$$

This is about 63 decibels.

- **Properties of Logarithms**

1. Logarithm rule 1: $\log(\mathrm{A}^t) = t\log(\mathrm{A})$
2. Logarithm rule 2: $\log(\mathrm{AB}) = \log(\mathrm{A}) + \log(\mathrm{B})$
3. Logarithm rule 3: $\log\left(\frac{\mathrm{A}}{\mathrm{B}}\right) = \log(\mathrm{A}) - \log(\mathrm{B})$

- **Example:** Suppose we have a population that is initially 500 and grows at a rate of 0.5% per month. How long will it take for the population to reach 800?

- **Solution:** The monthly percentage growth rate, $r = 0.005$.

The population size N after t months is:

$$N = \text{Initial value} \times (1 + r)^t = 500 \times 1.005^t$$

To find out when $N = 800$, solve the equation:

$$800 = 500 \times 1.005^t$$

Divide both sides by 500:

$$1.6 = 1.005^t$$

Apply the logarithm function to both sides and use rule 1:

$$\log 1.6 = \log(1.005^t) = t \log(1.005)$$

Dividing by log 1.005 gives:

$$t = \frac{\log 1.6}{\log 1.005} = 94.2 \text{ months}$$

The population reaches 800 in about 7 years and 10 months.

- **Solving exponential equations**

The solution for t of the exponential equation $A = B^t$ is:

$$t = \frac{\log \mathrm{A}}{\log \mathrm{B}}$$

- **Example:** An investment is initially \$5000 and grows by 10% each year. How long will it take the account balance to reach \$20,000?
- **Solution:** The balance B after t years:

$$B = \text{Initial value} \times (1 + r)^t = 5000 \times 1.1^t$$

To find when $B = \$20{,}000$,

solve $20{,}000 = 5000 \times 1.1^t$, or $4 = 1.1^t$ with $A = 4$ and $B = 1.1$ of exponential equation $A = B^t$.

$$t = \frac{\log \mathrm{A}}{\log \mathrm{B}} = \frac{\log 4}{\log 1.1} = 14.5 \text{ years}$$

Doubling Time and more

Suppose a quantity grows as an exponential function with a given base. The time *t* required to multiply the initial value by *K* is:

$$\text{Time required to multiply by } K \text{ is } t = \frac{\log K}{\log(\text{Base})}$$

The special case $K = 2$ gives the **doubling time:**

$$\text{Doubling time} = \frac{\log 2}{\log\,(\text{Base})}$$

- **Example:** Suppose an investment is growing by 7% each year. How long does it take the investment to double in value?

- **Solution:** The percentage growth is a constant, 7%, so the balance is an exponential function.
- The base $= 1 + r = 1.07$:

$$\text{Doubling time} = \frac{\log 2}{\log(\text{Base})}$$

$$= \frac{\log 2}{\log(1.07)} = 10.2 \text{ years}$$

- **Example:** Recall that carbon-14 has a half-life of 5770 years. Suppose the charcoal from an ancient campfire is found to contain only one-third of the carbon-14 of a living tree. How long ago did the tree that was the source of the charcoal die?

- **Solution:** Use $K = 1/3$, the base = half-life = 1/2:

$$\text{Time to multiply by } 1/3 \text{ is } t = \frac{\log K}{\log(\text{Base})} = \frac{\log(1/3)}{\log(1/2)} = 1.58.$$

Each half-life is 5770 years, the tree died $1.58 \times 5770 = 9116.6$ years ago.

Chapter 3 Linear and Exponential Changes:

Chapter Summary

- **Lines and linear growth:** What does a constant rate mean?
  - Understand linear functions and consequences of a constant growth rate.

    Recognizing and solve linear functions

    Calculate the growth rate or slope

    Interpolating and using the slope

    Approximate the linear data with trend lines

Chapter Summary

- **Exponential growth and decay:** Constant percentage rates
  - Understand exponential functions and consequences of constant percentage change.

    The nature of exponential growth

    Formula for exponential functions

    The rapidity of exponential growth

    Relating percentage growth and base

    Exponential decay

    Radioactive decay and half-life

Chapter Summary

- **Logarithmic phenomena:** Compressed scales
  - Understand the use if logarithms in compressed scales and solving exponential equations.
    - The Richter scale and interpolating change on the Richter scale
    - The decibel as a measure of sound
    - Solving exponential equations
    - Doubling time and more