

# Homotheties and incidences\*

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## Abstract

We consider problems involving rich homotheties in a set  $S$  of  $n$  points in the plane (that is, homotheties that map many points of  $S$  to other points of  $S$ ). By reducing these problems to incidence problems involving points and lines in  $\mathbb{R}^3$ , we are able to obtain refined and new bounds for the number of rich homotheties, and for the number of distinct equivalence classes, under homotheties, of  $k$ -element subsets of  $S$ , for any  $k \geq 3$ . We also discuss the extensions of these problems to three and higher dimensions.

## 1 Introduction

In this note we extend the analysis technique of Guth and Katz [7], which is based on the framework proposed by Elekes (as exposed, e.g., in [5]), to handle homotheties of point sets in the plane. The original technique in [7] derives a lower bound on the number of distinct distances determined by a set  $S$  of  $n$  points in  $\mathbb{R}^2$ . Equivalently, this can be thought of as obtaining a lower bound on the number of equivalence classes of pairs in  $S \times S$  under Euclidean motions. That is, two pairs  $(a, b)$ ,  $(a', b')$  are equivalent if there exists a rigid motion that maps  $a$  to  $a'$  and  $b$  to  $b'$  (which is the same as saying that  $|ab| = |a'b'|$ ).

In this note we consider the analogous problems that arise when we replace rigid motions by homotheties. Both rigid motions and homotheties have three degrees of freedom, and so can be represented as points in parametric 3-space. As we note here, Elekes's transformation can be applied in the context of homotheties too, and reduce problems that involve homotheties acting on a finite point set in the plane to problems that involve incidences between points and lines in three dimensions. The machinery developed in [7] then allows us to obtain various results concerning homotheties in the plane.

**Our results.** Specifically, we show that the number of *t-rich homotheties* in a set  $S$  of  $n$  points in the plane, namely, homotheties that map at least  $t$  points of  $S$  to other points of  $S$ ,

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is  $O\left(\frac{n^3}{t^2} + \frac{n^2\nu^2}{t^3}\right)$ , where  $\nu$  is the maximum size of a collinear subset of  $S$ . We also show, via a simple construction, that the second term cannot be improved. The upper bound is a consequence of a general incidence bound (given in Theorem 2.1 below) between any set of  $m$  homotheties (represented as points in  $\mathbb{R}^3$ ) and  $N$  lines of the form  $h_{p,q}$ , for pairs  $p, q \in S$ , where  $h_{p,q}$  is the locus of all homotheties that map  $p$  to  $q$  (which is indeed a line under a suitable and natural parameterization of homotheties—see below).

We then use this bound to obtain a lower bound on the number of pairwise non-homothetic  $k$ -tuples in  $S$ . The bound, given in Theorem 2.2, is  $\Omega(n^{k-1})$  for  $k \geq 4$ . For  $k = 3$  the bound depends on the parameter  $\nu$ : it is  $\Omega(n^2)$  if  $\nu = O(n/\sqrt{\log n})$ , and is  $\Omega\left(\frac{n^4}{\nu^2 \log n}\right)$  otherwise. As noted in [12], the bound is worst-case tight for  $k \geq 4$ ; it improves an earlier bound of  $\Omega(n^{k-2})$ . For  $k = 3$  an easy upper bound is  $O(n^2)$ , so our bound is optimal in this case too when  $\nu = O(n/\sqrt{\log n})$ .

**Background.** Problems involving distinct equivalence classes among  $k$ -tuples of point sets, under various transformations, have been posed and investigated quite some time ago. They are mentioned, e.g., in the monographs Brass et al. [2] and Pach and Sharir [12]. Many types of transformations have been considered, including rigid motions, homotheties, similarities, all the way to general affine and projective transformations. The simplest case is where  $k = 2$  and the transformations are rigid motions. In this case, as already noted, the question is how many distinct distances are determined by any set of  $n$  points in the plane. This problem has been almost completely settled in Guth and Katz [7], who derived the lower bound  $\Omega(n/\log n)$  for this quantity, almost matching Erdős’s upper bound  $O(n/\sqrt{\log n})$ .

Guth and Katz’s solution is based on an ingenious transformation due to Elekes (exposed, e.g., in [5]), which considers a standard representation of rigid motions in the plane as points in parametric 3-space, and maps each pair  $p, q$  of points in the input set  $S$  to a line  $h_{p,q}$ , which is the locus of all rigid motions that map  $p$  to  $q$ . (With the right choice of parameterization, given in [7],  $h_{p,q}$  is indeed a line.) By deriving new and sharper upper bounds on the number of incidences between points and lines in  $\mathbb{R}^3$ , Guth and Katz were able to obtain upper bounds on the number of  $t$ -rich rigid motions, namely motions that map at least  $t$  points of  $S$  to other points of  $S$ . These bounds, combined with another reduction of Elekes, yield the aforementioned lower bound on the number of distinct distances.

Rudnev [14] has later noticed that the same general approach can also handle the case  $k = 3$  for rigid motions, i.e., yield a lower bound on the number of pairwise non-congruent triangles determined by  $S$ , and also the case of  $k = 3$  under similarities. Rudnev obtained the lower bound  $\Omega(n^2)$  in the former case, and the lower bound  $\Omega(n^2/\log n)$  for the number of pairwise non-similar triangles.

However, as Rudnev was also aware of, the case of similarities has already been handled by Solymosi and Tardos [16], before the new algebraic machinery of Guth and Katz came into play. We will also comment on how to handle similarities, in the interest of completeness, but, similar to Rudnev, we do not see a way in which the algebraic technique, powerful as it is, can improve the bound in [16].

We therefore focus in this note on homotheties, which seem to have received less attention, and apply the algebraic machinery to obtain the aforementioned results. Problems

involving homotheties have been posed in several earlier works, including Brass [1], Brass et al. [2], and Pach and Sharir [12]. The earlier works (going back to van Kreveld and de Berg [10]) have mostly considered the “complementary” question of bounding the maximum number of  $k$ -subsets of a set  $S$  of  $n$  points in the plane that are homothetic to a given  $k$ -element “pattern”  $P$ . Elekes and Erdős [4] (see also Brass [1]) have shown that this quantity is  $\Theta(n^{1+1/k})$ , where  $k$  is the dimension of the rational affine hull of  $P$ . This becomes  $\Theta(n^2)$  for (only) one-dimensional patterns, and can be attained only under certain algebraicity assumptions, as shown in Laczkovich and Ruzsa [11]. The algorithmic issues of finding the homothetic copies of  $P$  in  $S$  are discussed in [1, 10].

Various open problems involving homotheties in three and higher dimensions are mentioned in [2, 12].

## 2 Homotheties in the plane

Each homothetic transformation of the plane (translation and scaling) has three degrees of freedom, and can therefore be represented parametrically as a point in  $\mathbb{R}^3$ . Let us use the representation  $(\xi, \eta, t)$ , where the homothety first scales the plane by  $t$ , with respect to the origin, and then translates it by the vector  $(\xi, \eta)$ . That is, the point  $(\xi, \eta, t)$  represents the transformation  $\tau_{\xi, \eta, t}(x) = tx + (\xi, \eta)$ , for  $x \in \mathbb{R}^2$ .

For a pair of points  $p, q \in \mathbb{R}^2$ , the locus in  $\mathbb{R}^3$  of all homotheties  $(\xi, \eta, t)$  that map  $p = (p_x, p_y)$  to  $q = (q_x, q_y)$  is the line  $h_{p,q}$ , given by the equations

$$\begin{aligned}\xi &= q_x - tp_x \\ \eta &= q_y - tp_y.\end{aligned}$$

Let  $S$  be a set of  $n$  points in the plane, and let  $L$  denote the collection of all  $n^2$  lines  $h_{p,q}$ , for  $p, q \in S$ .

A homothety  $(\xi, \eta, t)$  is incident to  $t$  lines  $h_{p_i, q_i}$  of  $L$ , for  $i = 1, \dots, t$ , if and only if it maps  $p_i$  to  $q_i$  for each  $i$ . In other words, we can reduce questions about homotheties acting on  $S$  to questions about incidences between points and lines in three dimensions.

The latter problem has been studied in Guth and Katz [7]. The bound that they obtain,<sup>1</sup> for the number of incidences between  $M$  points and  $N$  lines in  $\mathbb{R}^3$  is

$$I(M, N) = O\left(M^{1/2}N^{3/4} + M^{2/3}N^{1/3}s^{1/3} + M + N\right), \quad (1)$$

where  $s$  is the maximum number of input lines that lie in a common plane.

To apply (1) in our context, we estimate the parameter  $s$ , in our scenario, as follows. A plane  $\pi$  in  $\mathbb{R}^3$ , with equation  $\mathbf{x} \cdot \mathbf{v} = c$ , for a vector  $\mathbf{v} = (v_1, v_2, v_3)$  and a real  $c$ , contains the line  $h_{p,q}$  if

$$(q - tp) \cdot (v_1, v_2) + tv_3 = c$$

holds for every  $t$ . This happens if and only if  $p$  and  $q$  lie on the two respective parallel lines (in  $\mathbb{R}^2$ )

$$\begin{aligned}p \cdot (v_1, v_2) &= v_3 \\ q \cdot (v_1, v_2) &= c.\end{aligned}$$

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<sup>1</sup>Technically, this bound is implicit in, but directly follows from their results.

That is, in order to apply the incidence bound of Guth and Katz [7], we have to control configurations with many points lying on pairs of parallel lines. Actually, since the two lines do not have to be distinct, the parameter that we are after is the maximum number of points of  $S$  on any single line. We denote this quantity as  $\nu(S)$ , and put

$$\mu(S) := \max \{ |S \cap \ell| \cdot |S \cap \ell'| \mid \ell \text{ and } \ell' \text{ parallel lines} \} = \nu^2(S).$$

The preceding reasoning then implies that no plane in  $\mathbb{R}^3$  contains more than  $\mu = \mu(S)$  lines of  $L$ . Applying the bound in (1), we then get the following result.

**Theorem 2.1** *Let  $S$  be a set of  $n$  points in the plane, and put  $\nu = \nu(S)$ . Let  $L$  be the set of the  $n^2$  lines  $h_{p,q}$ , for  $p, q \in S$ , in  $\mathbb{R}^3$ , and let  $H$  be a set of  $m$  homotheties of the plane, represented as points in  $\mathbb{R}^3$ . Then the number of incidences between the points in  $H$  and the lines in  $L$  satisfies*

$$I(H, L) = O \left( m^{1/2} n^{3/2} + m^{2/3} n^{2/3} \nu^{2/3} + m + n^2 \right). \quad (2)$$

An analogous bound holds if we replace  $L$  by any subset  $L'$ ; the bound is then

$$I(H, L') = O \left( m^{1/2} |L'|^{3/4} + m^{2/3} |L'|^{1/3} \nu^{2/3} + m + |L'| \right).$$

In particular, the number  $M_{\geq t}$  of  $t$ -rich homotheties, namely those that map at least  $t$  points of  $S$  to other points of  $S$ , satisfies

$$M_{\geq t} = O \left( \frac{n^3}{t^2} + \frac{n^2 \nu^2}{t^3} + \frac{n^2}{t} \right) = O \left( \frac{n^3}{t^2} + \frac{n^2 \nu^2}{t^3} \right), \quad (3)$$

since the first term always dominates the third one. This bound follows in a standard manner by denoting by  $H_{\geq t}$  the set of  $t$ -rich homotheties, of cardinality  $m = M_{\geq t}$ , and by combining the bound in (2) with the inequality  $I(H_{\geq t}, L) \geq t M_{\geq t}$ . Of the remaining two terms, the first (resp., second) term dominates when  $\nu^2 \leq nt$  (resp.,  $\nu^2 \geq nt$ ).

**Lower bound.** We next show that the bound in (3) is tight in the worst case when  $\nu \geq \sqrt{nt}$ , that is, when the second term in (3) dominates. Assuming<sup>2</sup> that  $n \geq 16t$ , we also have  $\nu \geq 4t$ . Construct the set

$$S_0 = \{(i, 0) \mid i = 1, \dots, \nu\},$$

put  $t = n/\nu$ , create  $t$  translated copies of  $S_0$ , denoted as  $S_1, \dots, S_t$ , and let  $S$  be the union of these translated copies. We choose the translation vectors generically, to ensure that no non-horizontal line contains more than two points of  $S$ , and that, for any homothety of the plane that maps two horizontal lines that contain copies of  $S_0$  to two other such lines, one of the two source copies is such that none of its points are mapped to points of the corresponding target copy. Clearly,  $\nu(S) = \nu$ . To obtain a homothety that maps at least  $t$  points of  $S$  to other points of  $S$ , we choose a copy  $S_i$  of  $S_0$ , and choose an arithmetic progression in  $S_i$  of at least  $t$  elements. To do so, choose the difference of the sequence to

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<sup>2</sup>For  $n < t$  there are no  $t$ -rich homotheties. For  $t < n < 16t$  the upper bound in (3) is  $O(t)$ , and a matching lower bound, using a set of  $n$  equally spaced points on a line, is easy to derive (unless  $t = n - o(n)$ ).

be any integer  $1 \leq j < \nu/(2t)$ , and start the sequence at the  $i$ -th element, for any  $i < j$ . Denote the resulting sequence as  $A$ . Now pick another copy  $S_{i'}$  of  $S_0$ , and choose in it any pair of elements so that they are the first two elements of an arithmetic sequence  $B \subseteq S_{i'}$  of length at least  $t$ , and its difference is relatively prime to  $j$ . There are  $\Omega(\nu \cdot (\nu/t)) = \Omega(\nu^2/t)$  such pairs in  $S_{i'}$ , a bound that follows from standard properties of Euler's totient function (see, e.g., [3, Lemma 6.17] and [8]). We now map  $A$  to  $B$  by a homothety.

We claim that all these homotheties are distinct. Indeed, each such homothety is uniquely determined by the choice of the first two elements of  $A$  and the first two elements of  $B$ . Now, for two homotheties  $\tau_1, \tau_2$  to coincide, they must use the same source copy  $S_i$  and the same target copy  $S_{i'}$  of  $S_0$ . Assume that  $\tau_1$  (resp.,  $\tau_2$ ) is determined, as above, by  $p, q \in S_i$  and  $p', q' \in S_{i'}$  (resp., by  $r, s \in S_i$  and  $r', s' \in S_{i'}$ ); to simplify the reasoning, we use these symbols to refer also to the  $x$ -coordinates of the preimages of these points on  $S_0$ , with respect to the corresponding translations. Putting  $j = q - p$ ,  $j' = q' - p'$ ,  $a = s - r$ , and  $a' = s' - r'$ , we thus have  $\lambda = j'/j = a'/a$ , where  $\lambda$  is the scaling factor of the homothety  $\tau_1 = \tau_2$ . Since  $j, j'$  and  $a, a'$  are both relatively prime, it follows that  $a = j$  and  $a' = j'$ . We now claim that  $r = p$  too. If not, we have, by construction,  $p, r < j$ . Since  $p$  is mapped to  $p'$  and  $r$  to  $r'$ , we also have  $\lambda = \frac{r' - p'}{r - p}$ , which contradicts the fact that  $j$  and  $j'$  are relatively prime (as  $|r' - p'| < j'$  and  $|r - p| < j$ ). It now follows that  $(p, q, p', q') = (r, s, r', s')$ , and we thus conclude that distinct quadruples of this kind determine distinct homotheties, as claimed.

The number of such homotheties is thus at least

$$\Theta(t^2) \cdot \Theta\left(\left(\frac{\nu}{t}\right)^2 \cdot \frac{\nu^2}{t}\right) = \Theta\left(\frac{n^2}{\nu^2} \cdot \frac{\nu^4}{t^3}\right) = \Theta\left(\frac{n^2 \nu^2}{t^3}\right).$$

Unfortunately, we still do not know whether the bound (3) is tight also for the case  $\nu < \sqrt{nt}$ .

**Lower bound for distinct homothety classes.** Let  $S$  be a set of  $n$  points in the plane, and let  $k \geq 3$  be an integer parameter. Two ordered  $k$ -tuples  $(a_1, \dots, a_k), (b_1, \dots, b_k)$  are said to be *equivalent* under a homothety if there exists a homothety that maps  $a_i$  to  $b_i$ , for each  $i$ . We want to obtain a lower bound on the number of distinct equivalence classes of  $k$ -element subsets of  $S$  under homotheties. This is done using the following variant of Elekes's transformation.

Let  $Q$  denote the set of all  $2k$ -tuples  $(a_1, \dots, a_k, b_1, \dots, b_k)$  of elements of  $S$ , with the  $a_i$ 's all distinct and the  $b_i$ 's all distinct, such that  $(a_1, \dots, a_k)$  is equivalent to  $(b_1, \dots, b_k)$  under a homothety. Let  $x$  denote the number of distinct equivalence classes of  $k$ -tuples, and let  $E_1, \dots, E_x$  denote the classes themselves. Clearly, we have, by the Cauchy-Schwarz inequality,

$$|Q| = \sum_{i=1}^x \binom{|E_i|}{2} = \frac{1}{2} \sum_{i=1}^x |E_i|^2 - \frac{1}{2} \sum_{i=1}^x |E_i| \geq \frac{1}{2x} \left( \sum_{i=1}^x |E_i| \right)^2 - \frac{1}{2} \sum_{i=1}^x |E_i| = \Omega\left(\frac{n^{2k}}{x}\right),$$

where the last inequality holds if we assume that  $x \ll n^k$  (otherwise we get a better lower bound than the one we aim for—see the introduction and Theorem 2.2 below).

For an upper bound on  $|Q|$ , we note that every homothety that maps exactly  $t$  points of  $S$  to  $t$  other points generates

$$t(t-1) \cdots (t-k+1) \leq t^k$$

elements of  $Q$ , and we thus have

$$|Q| \leq \sum_{t \geq k} t^k M_t = O \left( k^k M_{\geq k} + \sum_{t \geq k+1} t^{k-1} M_{\geq t} \right),$$

where  $M_t$  (resp.,  $M_{\geq t}$ ) is the number of homotheties that map exactly  $t$  (resp., at least  $t$ ) points of  $S$  to other points of  $S$ .

Using the upper bound (3) on  $M_{\geq t}$ , we have

$$\begin{aligned} |Q| &= O \left( k^k \left( \frac{n^3}{k^2} + \frac{n^2 \nu^2}{k^3} \right) + \sum_{t \geq k+1} t^{k-1} \left( \frac{n^3}{t^2} + \frac{n^2 \nu^2}{t^3} \right) \right) \\ &= O \left( n^3 k^{k-2} + n^2 \nu^2 k^{k-3} + \sum_{t \geq k+1} \left( n^3 t^{k-3} + n^2 \nu^2 t^{k-4} \right) \right). \end{aligned}$$

For  $k = 3$  the sum is  $O(n^4 + n^2 \nu^2 \log n)$ . Combining this with the lower bound  $|Q| = \Omega(n^6/x)$ , we obtain

$$x = \begin{cases} \Omega(n^2) & \text{if } \nu = O(n/\sqrt{\log n}) \\ \Omega\left(\frac{n^4}{\nu^2 \log n}\right) & \text{otherwise.} \end{cases}$$

The situation is simpler for larger values of  $k$ , in which case we have

$$|Q| = O \left( n^{k+1} + n^{k-1} \nu^2 \right) = O \left( n^{k+1} \right),$$

implying that  $x = \Omega(n^{k-1})$ . That is, we have:

**Theorem 2.2** *The number of distinct equivalence classes of  $k$ -element subsets of a set  $S$  of  $n$  points in the plane, under homotheties, is  $\Omega(n^{k-1})$  for  $k \geq 4$ . For  $k = 3$  the lower bound depends on the maximum number  $\nu$  of points of  $S$  in any common line. It is  $\Omega(n^2)$  if  $\nu = O(n/\sqrt{\log n})$ , and is  $\Omega\left(\frac{n^4}{\nu^2 \log n}\right)$  otherwise.*

The theorem solves Problem 6.1 in Pach and Sharir [12]. As already noted, the bound is worst-case tight for  $k \geq 4$ , using a simple construction given in [12]; the previously best known lower bound was  $\Omega(n^{k-2})$  (see [12] for the easy argument). For  $k = 3$ , the bound is worst-case tight when  $\nu = O(n/\sqrt{\log n})$ . We leave it as an open problem to tighten the small remaining gap when  $\nu$  is larger.

**Joints.** A *joint* in  $L$  is a point (homothety) that is incident to at least three non-coplanar lines of  $L$ . By the preceding reasoning, if  $\tau$  is a homothety incident to the non-coplanar lines  $h_{p_i, q_i}$ ,  $i = 1, 2, 3$ , then  $p_1, p_2, p_3$  is a non-collinear triple, and so is the triple  $q_1, q_2, q_3$ , consisting of the respective images of  $p_1, p_2, p_3$  under  $\tau$ . The number of joints in  $L$  is  $O(|L|^{3/2}) = O(n^3)$  [6, 9, 13]. That is, there are at most  $O(n^3)$  homotheties that map at least three non-collinear points of  $S$  to other (non-collinear) points of  $S$ .

We mention this result because it does not depend on the parameter  $\nu(S)$ . Note that the bound in (3) is  $O(n^3)$  only for  $\nu(S) = O(n^{1/2})$ . Another way to interpret this finding is

that in order to get many  $t$ -rich homotheties we need  $S$  to contain many collinear points. Moreover, in the lower bound construction given above, all the homotheties that we construct map (at least  $t$ ) collinear points of  $S$  to other collinear points, and no other point of  $S$  is mapped to a point of  $S$ . The observation concerning joints, as just given, indicates that this is indeed unavoidable—the number of homotheties that map at least one non-collinear triple in  $S$  to another such triple is much smaller.

## 2.1 Homotheties in higher dimensions

Unlike the case of distinct distances (that is, of equivalence classes under rigid motions), Elekes’s transformation extends easily to higher dimensions in the case of homotheties. In  $\mathbb{R}^d$ , a homothety has  $d+1$  degrees of freedom, and can be represented by  $(\xi, t)$ , where  $t \in \mathbb{R}^+$  is the scaling factor and  $\xi \in \mathbb{R}^d$  is the translation vector. The locus  $h_{p,q}$  of homotheties (as points in  $\mathbb{R}^{d+1}$ ) that map a point  $p$  to another point  $q$  in  $\mathbb{R}^d$  is still a line, given by the system  $q = tp + \xi$  of  $d$  linear equations in  $d+1$  variables. Hence, the basic question that we face, analogous to the one studied in Theorem 2.1, is to estimate the number of incidences between points and lines in  $\mathbb{R}^{d+1}$ .

While much simpler than the corresponding transformation for rigid motions, this is far from being an easy problem, and it gets harder as  $d$  increases. So far the only known nearly tight bound for points and lines is in four dimensions (that is, for  $d = 3$ ), due to Sharir and Solomon [15]. For a set  $H$  of  $m$  points (homotheties in our case) and a set  $L$  of  $N$  lines (the  $n^2$  lines  $h_{p,q}$ , for  $p, q \in S$ , in our case), one has

$$I(H, L) = O\left(2^{c\sqrt{\log m}}(m^{2/5}N^{4/5} + m) + m^{1/2}N^{1/2}q^{1/4} + m^{2/3}N^{1/3}s^{1/3} + N\right), \quad (4)$$

for a suitable absolute constant  $c$ , provided that no 2-plane contains more than  $s$  lines of  $L$  and that no hyperplane or quadric contains more than  $q$  lines of  $L$ .

We will shortly use this bound to obtain an upper bound on the number of  $t$ -rich homotheties in a set  $S$  of  $n$  points in three dimensions, which is better than the one in (3) for  $t \ll n$  and for certain ranges of other parameters, discussed below. However, as it turns out, and perhaps surprisingly, this improved bound does not lead to an improved bound on the number of equivalence classes, under homotheties, of  $k$ -element subsets of  $S$ , for any  $k \geq 3$ . As we will show, the planar bounds, given in Theorem 2.2, are large enough, so that the improvement in the bound on the number of  $t$ -rich homotheties, an improvement that holds only when  $t \ll n$ , does not lead to a similar improvement in the number of equivalence classes.

Nevertheless, for the sake of its own interest, we proceed to bound the number of  $t$ -rich homotheties.

**Lines contained in planes, hyperplanes, or quadrics.** In order to apply the bound in (4), we first proceed to understand the geometric structure of the parameters  $q$  and  $s$  in (4).

For estimating  $s$ , we note that a 2-plane  $\pi$  is the intersection of two hyperplanes in  $\mathbb{R}^4$ ,

given by equations of the form

$$\begin{aligned}(\xi, t) \cdot (v_1, u_1) &= c_1 \\ (\xi, t) \cdot (v_2, u_2) &= c_2,\end{aligned}$$

for suitable vectors  $v_1, v_2 \in \mathbb{R}^3$  and scalars  $u_1, u_2, c_1, c_2$ . For a line  $h_{p,q}$  to be contained in  $\pi$ , we must have

$$\begin{aligned}(q - tp) \cdot v_1 + tu_1 &= c_1 \\ (q - tp) \cdot v_2 + tu_2 &= c_2,\end{aligned}$$

for every  $t$ , meaning that  $p$  and  $q$  must lie in the respective parallel lines (in  $\mathbb{R}^3$ )

$$\begin{aligned}q \cdot v_1 &= c_1 \\ q \cdot v_2 &= c_2, \quad \text{and} \\ p \cdot v_1 &= u_1 \\ p \cdot v_2 &= u_2.\end{aligned}$$

As in the planar case, these lines need not be distinct, so  $s = \nu^2$ , where  $\nu = \nu(S)$  is the maximum size of a collinear subset of  $S$ .

For estimating  $q$ , a simplified variant of the analysis just given shows that the maximum number of lines  $h_{p,q}$  that lie in a common hyperplane is  $\mu^2$ , where  $\mu = \mu(S)$  is the maximum number of points of  $S$  that lie in a common plane (in  $\mathbb{R}^3$ ). The situation is more involved for quadrics. Let  $Q$  be a quadric, whose equation is given by  $(\xi, t, 1)A(\xi, t, 1)^T = 0$ , for a suitable  $5 \times 5$  symmetric matrix  $A$ . Then  $h_{p,q}$  is contained in  $Q$  if

$$(q - tp, t, 1)A(q - tp, t, 1)^T = 0$$

for every  $t$ . That is, we must have

$$\begin{aligned}(q, 0, 1)A(q, 0, 1)^T &= 0 \\ (p, -1, 0)A(p, -1, 0)^T &= 0 \\ (p, -1, 0)A(q, 0, 1)^T &= 0.\end{aligned}$$

That is,  $p$  lies on a quadric  $Q_0$  in 3-space,  $q$  lies on another ‘‘similar’’ quadric (that has the same quadratic part as  $Q_0$ ), and  $p$  and  $q$  satisfy a bilinear equality induced by  $Q_0$  (the third equation given above). We can therefore bound, pessimistically, the number of lines  $h_{p,q}$  that lie on a quadric by  $\kappa^2$ , where  $\kappa = \kappa(S)$  is the maximum number of points of  $S$  that lie in a common quadric in 3-space (it looks like the actual bound should be smaller). That is, we have  $q \leq \max\{\mu^2, \kappa^2\}$ .

Substituting the bounds on  $s$  and  $q$  in (4), we get

$$I(H, L) = O\left(2^{c\sqrt{\log m}}(m^{2/5}N^{4/5} + m) + m^{1/2}N^{1/2}(\mu^{1/2} + \kappa^{1/2}) + m^{2/3}N^{1/3}\nu^{2/3} + N\right).$$

Arguing as above, with  $N = n^2$ , this implies that the number  $M_{\geq t}$  of  $t$ -rich homotheties satisfies

$$M_{\geq t} = O\left(\frac{2^{O(\sqrt{\log n})}n^{8/3}}{t^{5/3}} + \frac{n^2(\mu + \kappa)}{t^2} + \frac{n^2\nu^2}{t^3} + \frac{n^2}{t}\right). \quad (5)$$

Note that the first (resp., second) term in (5) is smaller than the planar counterpart term  $O(n^3/t^2)$  in (3) when  $t \ll n$  (resp., when  $\mu, \kappa \ll n$ ); the third and fourth terms in (5) are the same as in (3). That is, when  $t, \mu, \kappa \ll n$  we get a smaller bound on the number of  $t$ -rich homotheties in 3-space than we get in the plane.

**Lower bound for distinct homothety classes.** Let  $S$  be a set of  $n$  points in  $\mathbb{R}^3$ , and let  $k \geq 3$  be an integer parameter. As in the planar case, we estimate the number of distinct equivalence classes of  $k$ -element subsets of  $S$  under homotheties, via the set  $Q$  of all pairs of equivalent  $k$ -tuples. We have the same upper lower bound  $|Q| = \Omega(n^{2k}/x)$ , where  $x$  is the number of equivalence classes. For an upper bound on  $|Q|$ , we note, as before, that every homothety that maps exactly  $t$  points of  $S$  to  $t$  other points generates at most  $t^k$  elements of  $Q$ , implying that

$$|Q| \leq \sum_{t \geq k} t^k M_t = O\left(k^k M_{\geq k} + \sum_{t \geq k+1} t^{k-1} M_{\geq t}\right),$$

where  $M_t$  (resp.,  $M_{\geq t}$ ) is the number of homotheties that map exactly  $t$  (resp., at least  $t$ ) points of  $S$  to other points of  $S$ . We could have used the upper bound (5) on  $M_{\geq t}$  to estimate this expression, but, as can be easily verified, and as we have already forewarned, we do not get any improvement over the planar case. In fact, the bound is slightly worse because of the presence of the factor  $2^{O(\sqrt{\log n})}$  in (5).

We can get the same lower bound as in the plane by first arguing that the upper bound in (3) also holds in three (and in fact in any higher) dimensions. This is because a generic projection of the  $t$ -rich homotheties and of the lines  $h_{p,q}$  onto some generic 3-space has the properties that (i) incidences are preserved, (ii) no pair of lines and no pair of points (homotheties) have coinciding images, and (iii) no plane contains more than  $\nu^2$  lines. That genericity implies the first two properties is clear, and that it implies property (iii) requires a short and easy argument that we omit here.

Hence, using this reduction, we obtain that the lower bounds in Theorem 2.2 hold in any dimension  $d \geq 3$  too. They are worst-case tight for  $k \geq 4$ , and are tight for  $k = 3$  when  $\nu = O(n/\sqrt{\log n})$ .

**Similarities.** Going back to the plane, we remark that the case of similarities is also amenable to the technique of Elekes, which, in this case reduces the problem to incidence questions points and lines in the complex plane. Specifically, if we regard the real plane as the complex line  $\mathbb{C}$ , a similarity transformation in the plane is a linear transformation  $z \mapsto \xi z + \eta$ , for  $\xi, \eta \in \mathbb{C}$ , and vice versa. Indeed, multiplying by  $\xi$  represents rotation and scaling about the origin, and  $\eta$  is the subsequent translation vector. We thus represent similarities as points in  $\mathbb{C}^2$ . The locus  $h_{p,q}$  of all similarities that map  $p$  to  $q$  is the complex line  $p\xi + \eta = q$  in the  $\xi\eta$ -plane.

Using the extension of the Szemerédi-Trotter incidence bound to the complex plane, due to Tóth [17] and to Zahl [18], one can show, in a completely straightforward manner, that, for a set  $S$  of  $n$  points in the plane, the number of  $k$ -rich similarities (those that map at least  $k$  points of  $S$  to other points of  $S$ ) is  $O(n^4/k^3)$ , a bound already derived by Solymosi and Tardos [16], using a different (more “elementary”) technique, and also noted by Rudnev [14]. This in turn implies that the number of pairwise non-similar triangles determined by  $n$  points in the plane is  $\Omega(n^2/\log n)$ ; again, see [14, 16]. Exactly the same machinery can be used to derive lower bounds on the number of pairwise non-similar  $k$ -tuples determined by  $n$  points in the plane. Although they do not state it explicitly, the papers just cited could have also obtained this extension with their technique.

**A final note.** The geometric and algebraic structure of homotheties is much simpler than that of rigid motions (and of similarities). It is therefore somewhat surprising that the new bounds derived in this note have not been obtained earlier, by a more “direct” geometric approach, such as in [16]. While the application of the algebraic machinery to homotheties, as presented in this paper, is interesting and pleasing (to us), we honestly have no idea whether it is indeed necessary, and leave it as an interesting open problem to come up with more “elementary” proofs.

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