Conformal blocks and some of their uses

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These are notes from a spring 2026 course at Yale University. The course is now over, but the notes do not yet reflect everything that was discussed. In addition, they probably contain a lot of errors.

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1. Introduction

1.1. **The plan.** The course is about a subject which started in physics of 2-dimensional systems (partly string theory, but not only string theory). It has turned out to have a lot of applications, in diverse areas. [...]

1.2. **References.** In the main text I include a few references, for things I think need more explanation than I can provide; I didn't try to give correct attributions of the origin of the ideas and constructions we discuss, mostly because I am not capable. But let me add a few historical references here.

The idea of conformal block grows out of the structure of 2-dimensional conformal field theory, which I believe was first clearly elucidated in [1]. Vertex algebras are an abstraction of the structure of operator products in these theories, and in this form were originally defined in [2]. The affine and Virasoro vertex algebras, which are basic and important examples for our story, were defined in [3, 4]. The structure of modular tensor category arising from conformal field theory was first described (implicitly) in [5], and formalized in categorical language in [6]. [...]

2. Basic data

2.1. **Vertex algebras and modules.** The first basic datum of the course is a *vertex algebra* $\mathcal{V}$. A standard reference is [7]; our philosophy is close in spirit to [8]. For a concise set of lecture notes, see also [9].

Recall that an algebra is a vector space $\mathcal{A}$ with some additional structure: a linear map $\mathcal{A} \to \operatorname{End} \mathcal{A}$, obeying some conditions.

Similarly a vertex algebra is a vector space $\mathcal{V}$ with additional structure. We think of $a \in \mathcal{V}$ as objects which can be inserted at a point in the complex plane, giving an object $a(z)$ which varies analytically with z. But moreover we can insert multiple objects at multiple points of the plane. When we insert $a(z)$ and $b(0)$, and bring z close to 0, we think of the composite as being another element of $\mathcal{V}$ (or more precisely some completion of $\mathcal{V}$, associated to a small disc around 0).

Then for any $a \in \mathcal{V}$, $a(z)$ should provide an endomorphism $Y(a,z) \in \operatorname{End}\mathcal{V}$, depending analytically on z. We don't expect that $Y(a,z)$ exists at $z=0$; our motivation is from local defects in quantum field theory, and often there are singularities when two defects are placed at the same point. But still we should be able to expand in a series around $z=0$. So, for $b \in \mathcal{V}$ we will have an expansion

$$Y(a,z)b = \sum_{n\in\mathbb{Z}} z^{-n-1} a_{(n)} b$$

where each $a_{(n)} \in \operatorname{End}\mathcal{V}$. We require that this expansion is meromorphic as opposed to an essential singularity: i.e. $a_{(n)}b = 0$ for $n \gg 0$.

When we have $a,b,c \in \mathcal{V}$ we also can consider the combined object $a(z)b(w)c(0)$. This object should again lie in $\mathcal{V}$, depending analytically on z and w away from $z=0$ or $w=0$, except that it can also have a pole at $z=w$. This means that it may have different Laurent expansions around $z=w=0$ depending on whether $|w|<|z|$ or $|w|>|z|$.[1] We interpret these two expansions as $Y(a,z)Y(b,w)c$ and $Y(b,w)Y(a,z)c$ respectively.[2]

Now we can make the definition:

Definition 2.1 (Vertex algebra). A *vertex algebra* is a quadruple $(\mathcal{V}, |\Omega\rangle, T, Y)$ where $\mathcal{V}$ is a vector space, $|\Omega\rangle \in \mathcal{V}$, $T \in \operatorname{End}(\mathcal{V})$, and

$$Y(\,\cdot\,,z) : \mathcal{V} \to \operatorname{End}(\mathcal{V})[[z,z^{-1}]], \qquad Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1},$$

is a linear map, such that:

(1) $Y(|\Omega\rangle, z) = \mathbf{1}_{\mathcal{V}}$,

(2) $Y(a,z)|\Omega\rangle \in \mathcal{V}[[z]]$ and $Y(a,0)|\Omega\rangle = a$ (i.e. $a_{(n)}|\Omega\rangle = 0$ for $n \geq 0$ and $a_{(-1)}|\Omega\rangle = a$) ,

(3) $[T,\, Y(a,z)] = \partial_z\, Y(a,z)$ and $T|\Omega\rangle = 0$,

(4) For all $a,b,c \in \mathcal{V}$ there exists $F(z,w) \in \mathcal{V}[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$, such that $Y(a,z)Y(b,w)c$ and $Y(b,w)Y(a,z)c$ are the formal Laurent expansion of $F(z,w)$ in the regions $|w|<|z|$ and $|z|<|w|$ respectively. Or, equivalently: there exists N such that $(z-w)^N Y(a,z)Y(b,w)c = (z-w)^N Y(b,w)Y(a,z)c$.

Moreover we have a grading $\mathcal{V} = \bigoplus_{n\in\mathbb{N}} \mathcal{V}_n$, where each $\mathcal{V}_n$ is finite-dimensional, $\mathcal{V}_0 = \mathbb{C}|\Omega\rangle$, T has degree 1, and $a_{(n)}$ has degree $(\deg a) - n - 1$.

We will also sometimes use the notation $a_h = a_{((\deg a)+h-1)}$; then a_h has degree $-h$.

Some remarks:

- Some authors would allow more general notions of vertex algebra, and would call ours *CFT-type* vertex algebras; but this is broad enough for our purposes; our goal is mostly to explore the nicest examples.

[1] Recall the example

$$f(z,w) = \frac{1}{z-w} = \begin{cases} z^{-1}\sum_{n\geq 0}(w/z)^n & \text{for } |w|<|z|, \\ -w^{-1}\sum_{n\geq 0}(z/w)^n & \text{for } |w|>|z|. \end{cases}$$

These two Laurent expansions are different but related. Both of them give 1 if you multiply them by $z-w$. Thus their difference, in $\mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$, is annihilated by multiplication by $z-w$. This is an algebraic version of saying that they are Laurent expansions of the same function.

[2] The idea is that the operator closer to 0 acts first, and then the operator farther from 0 acts on the resulting object. We won't attempt to formalize this statement precisely now; for physicists it is an instance of *radial quantization*; it can be formalized in the language of *holomorphic factorization algebras*, see e.g. [10].

- Property (4) is sometimes called the "locality" axiom: it expresses the idea that singularities in $a(z)b(w)$ occur only at $z = w$, not anywhere else. We also say: $Y(a,z)$ and $Y(b,w)$ are *local* with respect to one another.
- Note that the grading implies that for each $a, b \in \mathcal{V}$ we have $a_{(n)}b = 0$ for all $n \gg 0$, i.e. $Y(a,z)b$ is a Laurent series in z.
- As you can perhaps see, a vertex algebra is a complicated object. They would be a little painful to construct by hand. We will describe an existence theorem below, which suffices for all our purposes.

Here are two key facts. Proper proofs can be found e.g. in [8] chapter 3.

Theorem 2.2 (Associativity of vertex algebras). For any $a, b \in \mathcal{V}$,

$$Y(a,z)Y(b,w) = Y(Y(a,z-w)b,w) = \sum_{n\in\mathbb{Z}} (z-w)^{-n-1} Y(a_{(n)}b,w) \ \in \operatorname{End}\mathcal{V}[[z^{\pm 1}, w^{\pm 1}]],$$

where in the middle and on the right we mean the expansion in the region $|z| > |w|$.

Theorem 2.2 expresses the idea that "$a(z)b(w)c(0)$" should stand for some well-defined analytic object which we can expand either in the region $|z| > |w| > 0$ or in the region $|w| > |z-w| > 0$.

Here is a proof sketch. First you show using (1)-(3) that

$$Y(a,z)Y(b,w)|\Omega\rangle = Y(Y(a,z-w)b,w)|\Omega\rangle\,.$$

(You use the integrated form of (3), namely $\mathrm{e}^{zT}Y(a,w)\mathrm{e}^{-zT} = Y(a,z+w)$, and combining that with (2), $\mathrm{e}^{zT}b = Y(b,z)|\Omega\rangle$.) You want to derive the same thing with $|\Omega\rangle$ replaced by any other vector, i.e. you want to insert a third object at 0. So act on both sides with $Y(c,u)$ on the left: this amounts to inserting that third object at large $|u|$. You get

$$Y(c,u)Y(a,z)Y(b,w)|\Omega\rangle = Y(c,u)Y(Y(a,z-w)b,w)|\Omega\rangle\,.$$

Then you want to move the $Y(c,u)$ to the right and substitute $u = 0$. You expect this should work, because of analytic continuation: the identity is true at large $|u|$, and u can avoid z and w on its path to $u = 0$, so there should be no problem. The way you realize this idea using Definition 2.1 is that you multiply both sides by a big power $(u-w)^N(u-z)^N$, then the commutation is allowed using axiom (4), and then set $u = 0$ and divide out the harmless $w^N z^N$.

Exercise 2.1. Work out the above proof sketch carefully.

Theorem 2.3 (Mode algebra of a vertex algebra). The modes $a_{(n)} \in \operatorname{End}\mathcal{V}$ for all $a \in \mathcal{V}$ span a Lie subalgebra $\operatorname{Lie}(\mathcal{V})$ of $\operatorname{End}(\mathcal{V})$, with

$$[a_{(m)},\, b_{(n)}] = \sum_{j\geq 0} \binom{m}{j} \left(a_{(j)}b\right)_{(m+n-j)}\,. \tag{2.1}$$

(Here we define the binomial coefficients by the usual formula, even when $m < 0$.)

Theorem 2.3 gives a useful way of thinking about vertex algebras concretely, and particularly their "local" behavior: it is determined by the corresponding Lie algebra of modes.

One way to think about Theorem 2.3 is as follows. Imagine that instead of formal series we were working with actual analytic objects in the complex plane. Then we could write[3]

$$\begin{aligned} a_{(m)}b_{(n)}c - b_{(n)}a_{(m)}c &= \left[\oint_{|z|=1}\oint_{|w|=1-\epsilon} - \oint_{|z|=1}\oint_{|w|=1+\epsilon}\right] z^m w^n\, F(z,w)\,\mathrm{d}z\mathrm{d}w \\ &= \oint_{|z|=1}\oint_{|w-z|=\epsilon} z^m w^n\, F(z,w)\,\mathrm{d}z\mathrm{d}w \\ &= \oint_{|z|=1} z^m \mathrm{d}z \left[\oint_{|w-z|=\epsilon} w^n\, F(z,w)\,\mathrm{d}w\right] \end{aligned}$$

Now, the polar part of $F(z,w)$ at $z=w$ is determined by Theorem 2.2: it gives $F(z,w) = \sum_{j\in\mathbb{Z}}(z-w)^{-j-1}Y(a_{(j)}b,w)c$. Using this we have

$$\begin{aligned} \oint_{|w-z|=\epsilon} w^n\, F(z,w)\,\mathrm{d}w &= \sum_{j\geq 0}\oint_{|w-z|=\epsilon} w^n (z-w)^{-j-1} Y(a_{(j)}b,w)c \\ &= \sum_{j\geq 0}\oint_{|w-z|=\epsilon} w^n (z-w)^{-j-1} \sum_{\ell\in\mathbb{Z}} w^{-\ell-1}(a_{(j)}b)_{(\ell)}c \\ &= \sum_{j\geq 0}\sum_{\ell\in\mathbb{Z}}\oint_{|w-z|=\epsilon} w^{n-\ell-1}(z-w)^{-j-1}(a_{(j)}b)_{(\ell)}c \\ &= \sum_{j\geq 0}\sum_{\ell\in\mathbb{Z}}\oint_{|t|=\epsilon} (z-t)^{n-\ell-1}t^{-j-1}(a_{(j)}b)_{(\ell)}c \\ &= \sum_{j\geq 0}\sum_{\ell\in\mathbb{Z}}\sum_{i\geq 0}\oint_{|t|=\epsilon} (-1)^i \binom{n-\ell-1}{i} z^{n-\ell-1-i}t^{i-j-1}(a_{(j)}b)_{(\ell)}c \end{aligned}$$

The contour-integral over t sets $i=j$. Inserting this in the full integral above, the outer integral over z sets $\ell = m+n-j$. Using $(-1)^j\binom{j-m-1}{j} = \binom{m}{j}$ this gives the desired formula (2.1). The actual proof in the world of power series, which I won't try to give here, is a sort of formal analog of this argument.

2.2. **Universal enveloping vertex algebras.** Many of the vertex algebras we consider are of a simple sort: they are *universal enveloping vertex algebras*.

Recall the universal enveloping algebra of a Lie algebra: given a Lie algebra $\mathfrak{g}$, $U(\mathfrak{g})$ is an associative algebra which is "freely generated by $\mathfrak{g}$," in the sense that there is a map $\mathfrak{g}\to U(\mathfrak{g})$ of Lie algebras, with a universal property: for any associative algebra A, any map $\mathfrak{g}\to A$ of Lie algebras factors as $\mathfrak{g}\to U(\mathfrak{g})\to A$. (So, for example, any Lie algebra representation of $\mathfrak{g}$ factors through an associative algebra representation of $U(\mathfrak{g})$.) $U(\mathfrak{g})$ is typically much bigger than $\mathfrak{g}$.

To define universal enveloping vertex algebras one does similarly. We first fix a finite-dimensional vector space R, with the idea that each $a\in R$ will give some $a(z)$. Then we give the singular part of the expansion $Y(a,z)Y(b,w)$, i.e. give products $a_{(n)}b$ for $n\geq 0$: this is the analogue of giving a Lie bracket. We organize them into the λ-*bracket* defined by

$$[a_\lambda b] = \sum_{j\geq 0}\frac{\lambda^j}{j!}a_{(j)}b$$

[3]We adopt the convention that $\oint$ always includes a factor $\frac{1}{2\pi\mathrm{i}}$.

or more analytically

$$Y([a_\lambda b], w) = \mathrm{Res}_{z=w} \, \mathrm{e}^{\lambda(z-w)} Y(a,z)Y(b,w) \, .$$

(Here the residue should be understood in the sense of Theorem 2.2.)

Giving this bracket amounts to the following structure:

Definition 2.4 (Lie conformal algebra). A *Lie conformal algebra* is a $\mathbb{C}[\partial]$-module R with a λ-bracket

$$[\cdot_\lambda \cdot] : R \otimes R \to R[\lambda] \, ,$$

expanded as

$$[a_\lambda b] = \sum_{j \geq 0} \frac{\lambda^j}{j!} a_{(j)} b \, ,$$

such that

$$[\partial a_\lambda b] = -\lambda \, [a_\lambda b] \, , \quad [a_\lambda \partial b] = (\lambda + \partial) \, [a_\lambda b] \, , \quad [a_\lambda b] = - \, [b_{-\lambda-\partial} a] \, ,$$
$$[a_\lambda \, [b_\mu c]] - [b_\mu \, [a_\lambda c]] = \Big[[a_\lambda b]_{\lambda+\mu} \, c \Big] \, .$$

Any vertex algebra is a Lie conformal algebra (just as any associative algebra is a Lie algebra), and we have:

Proposition 2.5 (Existence of universal enveloping vertex algebra). Suppose R is a Lie conformal algebra. Then there exists a vertex algebra $U(R)$ and a morphism $R \to U(R)$ of Lie conformal algebras, such that for any vertex algebra $\mathcal{V}$, any morphism of Lie conformal algebras $R \to \mathcal{V}$ factors as $R \to U(R) \to \mathcal{V}$.

Here are three main examples:

(1) The *Heisenberg algebra* $\mathcal{V} = \mathrm{Heis}$. Here the Lie conformal algebra is

$$R = (\mathbb{C}[\partial] \cdot J) \oplus (\mathbb{C} \cdot \mathbf{1})$$

where $\partial \mathbf{1} = 0$. The J-J operator product is

$$J(z)J(w) = \frac{1}{(z-w)^2} + \mathrm{reg}$$

i.e. $[J_\lambda J] = \lambda \mathbf{1}$, while $\mathbf{1}$ is central. The mode algebra is spanned by $J_{(n)}$ $(n \in \mathbb{Z})$ and $\mathbf{1} = \mathbf{1}_{(-1)}$, with

$$[J_{(m)}, J_{(n)}] = m \delta_{m+n,0} \mathbf{1} \, .$$

(2) For any complex Lie algebra $\mathfrak{g}$ with invariant bilinear form κ and $k \in \mathbb{C}$, one has the *affine Kac-Moody algebra* $\mathcal{V} = V^k(\mathfrak{g})$. Here

$$R = (\mathbb{C}[\partial] \cdot \mathfrak{g}) \oplus (\mathbb{C} \cdot \mathbf{1})$$

with operator products of the form

$$J^X(z)J^Y(w) = \frac{k\kappa(X,Y)}{(z-w)^2} + \frac{J^{[X,Y]}(w)}{z-w} + \mathrm{reg} \tag{2.2}$$

i.e. $\left[J^X_\lambda J^Y\right] = \lambda k \kappa(X,Y) \mathbf{1} + J^{[X,Y]}$. The corresponding mode algebra is

$$[J^X_{(m)}, J^Y_{(n)}] = J^{[X,Y]}_{(m+n)} + mk\kappa(X,Y)\delta_{m+n,0} \mathbf{1} \, .$$

Note that the zero modes $J^X_{(0)}$ have

$$[J^X_{(0)}, J^Y_{(0)}] = J^{[X,Y]}_{(0)}$$

so they form a copy of $\mathfrak{g}$ inside $\mathrm{Lie}(\mathcal{V})$. Also note the Heisenberg vertex algebra is a special case, where $\mathfrak{g} = \mathfrak{gl}_1$ with a single generator X, $\kappa(X,X) = 1$ and $k = 1$.

(3) For any $c \in \mathbb{C}$ one has the *Virasoro algebra* Vir^c. It is generated by a defect T, with an expansion of the form

$$T(z)T(w) = \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \mathrm{reg}$$

i.e. $[T_\lambda T] = \lambda^3 \frac{c}{12}\mathbf{1} + (2\lambda + \partial)T$. The corresponding mode Lie algebra is

$$[T_{(m)}, T_{(n)}] = (m-n)\, T_{(m+n-1)} + \frac{c}{12}\, m(m-1)(m-2)\, \delta_{m+n,2}\mathbf{1}\,,$$

or after defining $L_m = T_{(m+1)}$ to match the usual convention,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}\,(m+1)(m)(m-1)\,\delta_{m+n,0}\mathbf{1}\,. \tag{2.3}$$

Exercise 2.2. Derive the Lie algebra (2.3) from the given $[T_\lambda T]$ and (2.1).

In all these examples $\mathcal{V} = U(R)$ where $R/(\mathbb{C}\cdot\mathbf{1})$ is free over $\mathbb{C}[\partial]$. In this situation we have a PBW-type basis:

Proposition 2.6 (PBW basis for universal enveloping vertex algebra). Let R be a Lie conformal algebra with a subalgebra T annihilated by ∂, such that R/T is free as a $\mathbb{C}[\partial]$-module. Then fix an ordered $\mathbb{C}[\partial]$-basis $(a^i)_{i\in I}$ of R/T. Then the vectors

$$a^{i_1}_{(-n_1-1)}\, a^{i_2}_{(-n_2-1)} \cdots a^{i_r}_{(-n_r-1)}\, |\Omega\rangle,$$

with $r \geq 0$, $n_1 \geq n_2 \geq \cdots \geq n_r \geq 0$, and $i_k \leq i_{k+1}$ whenever $n_k = n_{k+1}$, form a $\mathbb{C}$-basis of $U(R)$.

Equivalently, if $\mathfrak{L}$ is the mode Lie algebra on R/T, and $\mathfrak{L}_{<0} := \mathrm{span}\{a_{(n)} : a \in R,\ n < 0\}$, then the multiplication map induces a vector-space isomorphism

$$U(\mathfrak{L}_{<0}) \xrightarrow{\sim} U(R), \qquad u \mapsto u\,|\Omega\rangle.$$

Example 2.7. Suppose $\mathcal{V} = \mathrm{Heis}$. The above theorem says that $\mathcal{V}_n$ has a basis indexed by unordered partitions of n, e.g. $\mathcal{V}_4$ is spanned by the vectors $J^4_{(-1)}|\Omega\rangle$, $J_{(-2)}J^2_{(-1)}|\Omega\rangle$, $J^2_{(-2)}|\Omega\rangle$, $J_{(-3)}J_{(-1)}|\Omega\rangle$, $J_{(-4)}|\Omega\rangle$.

Example 2.8. Suppose $\mathcal{V} = V^k(\mathfrak{g})$. Then $\mathcal{V}_0$ is spanned by $|\Omega\rangle$, $\mathcal{V}_1$ is spanned by $J^X_{(-1)}|\Omega\rangle$ for all $X \in \mathfrak{g}$ (i.e. $\mathcal{V}_1 \simeq \mathfrak{g}$), $\mathcal{V}_2$ by $J^X_{(-1)}J^Y_{(-1)}|\Omega\rangle$ and $J^X_{(-2)}|\Omega\rangle$, and so on. Each level $\mathcal{V}_n$ is a representation of $\mathfrak{g}$, through the action of the zero modes $J^X_{(0)}$. $\mathcal{V}_0 \simeq \mathbf{triv}$, $\mathcal{V}_1 \simeq \mathfrak{g}$, $\mathcal{V}_2 \simeq \mathfrak{g} \oplus \mathrm{Sym}^2\,\mathfrak{g}$, and so on.

A useful way to think about this answer, at least informally (for $\mathcal{V} = \mathrm{Heis}$): we started out with a single object J, which we thought of as a "thing which can be inserted at a point $z \in \mathbb{C}$". Then we asked, what are the things which appear in the Laurent expansion when we bring two J's close together, and then iterate? These other things can still be inserted at a point, while still behaving meromorphically with respect to J. The answer includes J but also many other things,

all constructed by taking derivatives of J, or taking contour integrals of one object around another, and iterating. A basis for them is provided by iterated circular contour-integrals

$$\oint \cdots \oint z_1^{n_1} \cdots z_k^{n_k} J(z_1) \cdots J(z_k)\,.$$

It's similar for the other universal enveloping vertex algebras: we can think of them as spanned by iterated contour-integrals of a set of generating fields.

2.3. **Some vertex operators.** It is interesting to ask what are the vertex operators $Y(a,z)$ associated to general elements $a \in \mathcal{V}$. We know that

$$Y(a_{(-1)}|\Omega\rangle, z) = Y(a,z) = \sum_n a_{(n)} z^{-n-1}\,.$$

Using the translation axiom we can similarly get

$$Y(a_{(-m)}|\Omega\rangle, z) = \frac{\partial_z^{m-1}}{(m-1)!} Y(a,z)\,.$$

Now what about something like $Y(J_{(-1)}J_{(-1)}|\Omega\rangle, z)$? $Y(J_{(-1)}J_{(-1)}|\Omega\rangle, z)$ should be a local field which creates the state $J_{(-1)}J_{(-1)}|\Omega\rangle$ at z, i.e. it should be $\oint_z (w-z)^{-1} J(w)J(z)\mathrm{d}w$. This is an instance of a regularized product between two fields, called *normal ordered product*:

Definition 2.9 (Normally ordered product). For $a, b \in \mathcal{V}$ we define

$$:ab: = a_{(-1)}b \in \mathcal{V}\,.$$

We can also write it in another way:

$$\begin{aligned} :JJ:(z) &= \oint_w (z-w)^{-1} J(z)J(w)\,\mathrm{d}z \\ &= \oint_w (z-w)^{-1} \left(J(z)J(w) - \frac{\mathbf{1}}{(z-w)^2} \right) \mathrm{d}z = \lim_{w\to z} \left(J(z)J(w) - \frac{\mathbf{1}}{(z-w)^2} \right)\,. \end{aligned}$$

So $:JJ:(z)$ can be obtained by starting with $J(w)J(z)$, deleting "by hand" the singular term $\frac{\mathbf{1}}{(z-w)^2}$, and taking $w \to z$.

Here is the reason why it is called "normal ordering":

Proposition 2.10. Define $Y(a,z)_< = \sum_{n<0} a_{(n)} z^{-n-1}$ and $Y(a,z)_\geq = \sum_{n\geq 0} a_{(n)} z^{-n-1}$. Then

$$Y(:ab:,z) = Y(a,z)_< Y(b,z) + Y(b,z) Y(a,z)_\geq$$

In terms of modes,

$$:ab:_{(n)} = \sum_{k+\ell=n-1} \begin{cases} a_{(k)}b_{(\ell)} & \text{if } k<0, \\ b_{(\ell)}a_{(k)} & \text{if } k \geq 0 \end{cases}$$

So, for instance,

$$:JJ:_{(-1)} = J_{(-1)}J_{(-1)} + 2J_{(-2)}J_{(0)} + 2J_{(-3)}J_{(1)} + \cdots$$

Note that this sum actually makes sense in $\mathrm{End}\,\mathcal{V}$, because $J_{(n)}a = 0$ for $n \gg 0$.

The analytic explanation of this formula goes as follows. As we just discussed we expect to obtain $:ab:(w)$ by expanding $a(z)b(w)$ in $(z-w)$ at fixed w, subtracting all the singular terms, and then setting $z-w=0$. Let's see how this works out in Laurent expansions, choosing the

region $|z| > |w|$. Using Theorem 2.2 the singular part of $Y(a,z)Y(b,w)$ is the $|z| > |w|$ expansion of

$$\sum_{n\geq 0}(z-w)^{-n-1}Y(a_{(n)}b,w)$$

Now recall that $[Y(a,z),Y(b,w)]$ is the *difference* between the two ways of expanding that sum. Moreover, the two expansions are easy to distinguish: the expansion in $|z| > |w|$ involves only negative powers of z, while the one in $|z| < |w|$ involves only nonnegative powers. It follows that $[Y(a,z)_{\geq}, Y(b,w)]$ is the expansion in the region $|z| > |w|$. In other words, subtracting the singular terms amounts to taking

$$Y(:ab:,w) = \left(Y(a,z)Y(b,w) - [Y(a,z)_{\geq}, Y(b,w)]\right)|_{z=w}$$

which is the desired formula.

Once we know this formula we can use it iteratively to determine all other vertex operators. For instance:

Exercise 2.3. Calculate the mode expansion of $Y(J_{(-1)}J_{(-1)}J_{(-1)}|\Omega\rangle, z)$.

Exercise 2.4. Calculate the mode expansion of $Y(J_{(-2)}J_{(-1)}|\Omega\rangle, z)$.

2.4. **Modules for vertex algebras.**

Definition 2.11 (Module for a vertex algebra). A (weak) $\mathcal{V}$*-module* is a vector space $\mathcal{R}$ and an operation

$$Y^{\mathcal{R}}(\,\cdot\,,z) : \mathcal{V} \to \mathrm{End}(\mathcal{R})[[z,z^{-1}]], \qquad Y^{\mathcal{R}}(a,z) = \sum_{n\in\mathbb{Z}} a^{\mathcal{R}}_{(n)} z^{-n-1},$$

such that

(1) For each $a \in \mathcal{V}$, $\chi \in \mathcal{R}$ we have $a^{\mathcal{R}}_{(n)}\chi = 0$ for all $n \gg 0$,

(2) $Y^{\mathcal{R}}(|\Omega\rangle, z) = \mathbf{1}_{\mathcal{R}}$,

(3) $Y^{\mathcal{R}}(Ta,z) = \partial_z Y^{\mathcal{R}}(a,z)$,

(4) For all $a,b \in \mathcal{V}$ and $\chi \in \mathcal{R}$ there exists $F(z,w) \in \mathcal{R}[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$, such that $Y^{\mathcal{R}}(a,z)Y^{\mathcal{R}}(b,w)\chi$ and $Y^{\mathcal{R}}(b,w)Y^{\mathcal{R}}(a,z)\chi$ are the formal Laurent expansion of $F(z,w)$ in the regions $|w| < |z|$ and $|z| < |w|$ respectively.

Definition 2.12 (Admissible module). A $\mathcal{V}$-module $\mathcal{R}$ is *admissible* if $\mathcal{R} = \bigoplus_n \mathcal{R}_n$ (warning: the grading is not necessarily by integers), where $a^{\mathcal{R}}_{(n)}$ acts with degree $(\deg a) - n - 1$.

2.5. **Riemann surfaces.**

The second main datum of the course is a *Riemann surface* C, i.e. a complex manifold of complex dimension 1. We usually take C to be compact. Two fundamental examples are

(1) the *Riemann sphere* $C = \mathbb{CP}^1$, covered by two patches with coordinates z and z', related by $z' = 1/z$.

(2) the *torus* $C = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$, depending on the parameter $\tau \in \mathbb{C}$ with $\mathrm{Im}\,\tau > 0$.

We also often include marked points p_i on C, each equipped with a local coordinate around it. Each marked point p_i will be labeled by a $\mathcal{V}$-module $\mathcal{R}_i$.

We often drop the data of the marked points and the representations from the notation, just writing the whole object as C.

3. Chiral correlation functions and conformal blocks

Now we are ready to "put $\mathcal{V}$ on the Riemann surface C." The way this is done for a general $\mathcal{V}$ takes a while to appreciate. To get some examples in our hand quickly, we will first specialize to the case of affine vertex algebras, and further to the case of a curve C with no marked points. Then we will treat more general situations.

Before defining conformal blocks we consider correlation functions. I think that these are actually the more important objects (and in good cases they are the same thing as conformal blocks).

3.1. Chiral correlation functions without marked points.

Definition 3.1 (Chiral correlation system for $V^k(\mathfrak{g})$, without marked points). Suppose given a Riemann surface C. A *chiral correlation system* for $V^k(\mathfrak{g})$ on C is a rule that assigns an element

$$\left\langle \prod_{j=1}^{m} J^{X_j}(q_j) \right\rangle \in \bigotimes_{j=1}^{m} T^*_{q_j} C$$

for every $m \geq 0$, $q_j \in C$, which:

- is linear in each X_i,
- is symmetric under the interchange of the $J^{X_i}(q_i)$,
- is meromorphic in the q_i, and holomorphic away from loci where some $q_i = q_j$,
- satisfies the relations (2.2) around $q_i = q_j$, i.e. the expansion of $\langle J^X(q) J^Y(q') \cdots \rangle$[4] around $q = q'$ is

$$\langle J^X(q) J^Y(q') \cdots \rangle = \frac{k\kappa(X,Y)\langle \cdots \rangle \mathrm{d}z(q)\mathrm{d}z(q')}{(z(q)-z(q'))^2} + \frac{\langle J^{[X,Y]}(q') \cdots \rangle \mathrm{d}z(q)}{z(q)-z(q')} + \mathrm{reg}\,. \tag{3.1}$$

Exercise 3.1. Prove that the condition (3.1) is coordinate-invariant.

Exercise 3.2. Prove that chiral correlation systems form a vector space over $\mathbb{C}$.

Example 3.2 (Chiral correlation systems for the Heisenberg vertex algebra on the sphere). Here is a baby example.

Suppose $\mathcal{V} = \mathrm{Heis} = V^1(\mathfrak{gl}_1)$, and $C = \mathbb{CP}^1$. Then we can construct a chiral correlation system as follows. Let σ denote a division of $\{1, \ldots, k\}$ into 2-element subsets $(\sigma_{i1}, \sigma_{i2})$ with $i = 1, \ldots, \frac{k}{2}$. Then we can determine a chiral correlation system by

$$\langle J(z_1) \cdots J(z_k) \rangle = \sum_{\sigma} \prod_{i=1}^{\frac{k}{2}} \frac{\mathrm{d}z_{\sigma_{i1}} \mathrm{d}z_{\sigma_{i2}}}{(z_{\sigma_{i1}} - z_{\sigma_{i2}})^2}$$

[4] Here and below, we use the notation $\cdots$ to mean some arbitrary allowed insertion in a correlation function; e.g. here $\cdots$ stands for an expression of the form $\prod_{i=1}^{m-2} J^{X_i}(q_i)$; in any equation where $\cdots$ appears multiple times, the understanding is that it stands for the same expression each time.

so e.g.

$$\begin{aligned}
\langle\rangle &= 1,\\
\langle J(z_1)\rangle &= 0,\\
\langle J(z_1)J(z_2)\rangle &= \frac{1}{(z_1-z_2)^2}\mathrm{d}z_1\mathrm{d}z_2,\\
\langle J(z_1)J(z_2)J(z_3)\rangle &= 0,\\
\langle J(z_1)J(z_2)J(z_3)J(z_4)\rangle &= \left(\frac{1}{(z_1-z_2)^2(z_3-z_4)^2}+\frac{1}{(z_1-z_3)^2(z_2-z_4)^2}+\frac{1}{(z_1-z_4)^2(z_2-z_3)^2}\right)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z
\end{aligned}$$

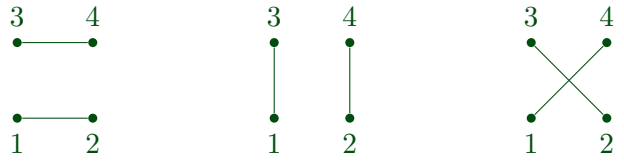
We can make a diagrammatic representation of this answer as follows. Let

$$B(z,w)=\frac{\mathrm{d}z\,\mathrm{d}w}{(z-w)^2}.$$

Then

$$\langle J(z_1)\cdots J(z_k)\rangle=\sum_{\Gamma}\prod_{\{i,j\}\in E(\Gamma)}B(z_i,z_j),$$

where the sum is over Feynman diagrams Γ whose vertices are $\{1,\dots,k\}$ and whose edges form a *perfect matching* (i.e. every vertex has valence 1). Here are the three diagrams which contribute when $k=4$.



Exercise 3.3. Prove that this rule indeed defines a chiral correlation system.

Exercise 3.4. Prove that every chiral correlation system is a scalar multiple of the one we just described. Thus, the space of chiral correlation systems for this $(C,\mathcal{V})$ is 1-dimensional.

Example 3.3 (Chiral correlation systems for the Heisenberg vertex algebra on the torus). Take $C=\mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$ and $\mathcal{V}=\mathrm{Heis}$. We can construct a chiral correlation system depending on a parameter $a\in\mathbb{C}$ as follows. Let $\wp_\tau(z)$ be the Weierstrass function: this is a meromorphic function obeying

$$\wp_\tau(z+1)=\wp_\tau(z),\qquad \wp_\tau(z+\tau)=\wp_\tau(z),\qquad \wp_\tau(z)=\wp_\tau(-z)\,.$$

$\wp_\tau$ descends to a function on $C=\mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$, with a single pole at $z=0$, with Laurent expansion

$$\wp_\tau(z)=\frac{1}{z^2}+\mathrm{reg}\,.$$

Let

$$B(z,w)=\wp_\tau(z-w)\,\mathrm{d}z\,\mathrm{d}w\,,\qquad \eta(z)=a\,\mathrm{d}z\,.$$

Then

$$\langle J(z_1)\cdots J(z_k)\rangle = \sum_{\Gamma} \prod_{\{i,j\}\in E(\Gamma)} B(z_i,z_j) \prod_{i\in V_0(\Gamma)} \eta(z_i),$$

where the sum is over Feynman diagrams Γ whose vertices are $\{1,\dots,k\}$, with every vertex either 0-valent or 1-valent. $V_0(\Gamma)$ is the set of 0-valent vertices.

For example,

$$\langle J(z)\rangle = a\,\mathrm{d}z,$$
$$\langle J(z_1)J(z_2)J(z_3)\rangle = (a\wp_\tau(z_1-z_2)+a\wp_\tau(z_1-z_3)+a\wp_\tau(z_2-z_3)+a^3)\,\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\,.$$

Exercise 3.5. Prove that this rule indeed defines a chiral correlation system.

Exercise 3.6. Prove that the chiral correlation systems we just described, for different values of $a\in\mathbb{C}$, are all linearly independent. Thus in this case the space of chiral correlation systems is infinite-dimensional.

Example 3.4 (Chiral correlation systems for an affine vertex algebra on the sphere). Here is a construction of chiral correlation functions, originally given in [4]. [check factors, signs] Define

$$W(i_1,\dots,i_m) = \frac{k\,\kappa(X_1,\mathrm{ad}_{X_2}\mathrm{ad}_{X_3}\cdots\mathrm{ad}_{X_{m-1}}(X_m))}{(z_{i_1}-z_{i_2})(z_{i_2}-z_{i_3})\cdots(z_{i_m}-z_{i_1})}\,\mathrm{d}z_{i_1}\cdots\mathrm{d}z_{i_m}\,.$$

Then the correlator is a sum over decompositions of $\{1,\dots,n\}$ into disjoint cycles with length $m\geq 2$:

$$\left\langle \prod_{r=1}^{n} J^{X_r}(z_r)\right\rangle = \sum_{\{1,\dots,n\}=C_1\sqcup\cdots\sqcup C_s} \prod_{\alpha=1}^{s}\left(\frac{1}{|C_\alpha|}\sum_{\text{orderings }\mathbf{i}\text{ of }C_\alpha} W(\mathbf{i})\right).$$

For example,

$$\langle J^{X_1}(z_1)J^{X_2}(z_2)J^{X_3}(z_3)\rangle = \frac{k\,\kappa(X_1,[X_2,X_3])}{(z_1-z_2)(z_2-z_3)(z_3-z_1)}\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3$$

and

$$\langle J^{X_1}(z_1)J^{X_2}(z_2)J^{X_3}(z_3)J^{X_4}(z_4)\rangle =$$
$$\left(\left(\frac{k\,\kappa(X_1,[X_2,[X_3,X_4]])}{(z_1-z_2)(z_2-z_3)(z_3-z_4)(z_4-z_1)}+\cdots\right)+\left(\frac{k^2\,\kappa(X_1,X_2)\kappa(X_3,X_4)}{(z_1-z_2)^2(z_3-z_4)^2}+\cdots\right)\right)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_4$$

where each $\cdots$ is a sum over two similar terms.

Exercise 3.7. Prove that this rule indeed defines a chiral correlation system.

Exercise 3.8. Prove that every chiral correlation system is a scalar multiple of the one we just described. Thus, the space of chiral correlation systems for this $(C,\mathcal{V})$ is 1-dimensional.

3.2. **Including marked points with the vacuum module.** Suppose we have a chiral correlation system for $V^k(\mathfrak{g})$ as above. Choose points $p_1, \dots, p_n \in C$ and fix a local coordinate z_i centered at each p_i. Then we can upgrade the correlation functions to permit arbitrary insertions of elements of $\mathcal{V}$, instead of just J^X: i.e. we can consider objects

$$\langle \prod_{r=1}^{k} J^{X_r}(q_r) \prod_{i=1}^{n} \cdot(p_i) \rangle : \left(\bigotimes_{i=1}^{n} \mathcal{V} \right) \to \left(\otimes_{i=1}^{k} T^*_{q_i} C \right)$$

determined by two conditions:

(1) Insertions of $|\Omega\rangle$ can be erased, i.e.

$$\langle \prod_{r=1}^{k} J^{X_r}(q_r) \prod_{i=1}^{n} \cdot(p_i) \rangle (|\Omega\rangle, a_2, \dots, a_n) = \langle \prod_{r=1}^{k} J^{X_r}(q_r) \prod_{i=2}^{n} \cdot(p_i) \rangle (a_2, \dots, a_n)$$

(2) The Laurent expansion around $q = p$ is determined by the action of J^X on $\mathcal{V}$,

$$\langle \cdots J^X(q) a(p) \rangle = \sum_n z(q)^{-n-1} \langle \cdots (J^X_{(n)} a)(p) \rangle$$

By contour-integration this formula determines the terms on the RHS. So this gives a way of inductively constructing up these correlation functions starting from the chiral correlation system. There are two extreme cases:

- $n = 0$ recovers the original chiral correlation system;
- $k = 0$ gives a map $(\otimes_{i=1}^{n} \mathcal{V}) \to \mathbb{C}$. That data-type doesn't involve the Riemann surface C, but C still enters implicitly, through constraints which this linear map obeys.

Example 3.5. Take $C = \mathbb{CP}^1$. Use the standard coordinates z and $w = 1/z$, and choose the marked points to be $z = 0$ and $w = 0$, i.e. north and south pole. Then the chiral correlation system we discussed above gives a map $\Phi : \mathcal{V} \otimes \mathcal{V} \to \mathbb{C}$. What is that map? First, we have

$$\Phi(|\Omega\rangle, |\Omega\rangle) = \langle 1 \rangle = 1 \,.$$

Second, by contour-deformation we have[5]

$$\Phi(J^X_{(n)} a, b) = \oint_{z=0} z^n \langle b(\infty) J^X(z) a(0) \rangle = - \oint_{w=0} w^{-n} \langle b(\infty) J^X(w) a(0) \rangle = -\Phi(a, J^X_{(-n)} b).$$

(Or equivalently, the sum of residues of the global meromorphic 1-form $\langle z^n b(\infty) J^X(z) a(0) \rangle$ vanishes.)

These two facts are enough to iteratively determine Φ. For example, on the grade-1 subspace:

$$\Phi(J^X_{(-1)}|\Omega\rangle, J^Y_{(-1)}|\Omega\rangle) = -\Phi(|\Omega\rangle, J^X_{(1)} J^Y_{(-1)}|\Omega\rangle) = -\Phi(|\Omega\rangle, k\kappa(X,Y)|\Omega\rangle) = -k\kappa(X,Y) \,.$$

Given Φ we could actually reconstruct the whole chiral correlation system, by summing up Laurent series, e.g.

$$\langle J^X(z) J^Y(w) \rangle = \sum_{n,m \in \mathbb{Z}} z^{-n-1} w^{-m-1} \Phi(|\Omega\rangle, J^X_{(n)} J^Y_{(m)} |\Omega\rangle) \text{ for } |z| > |w| \,.$$

[5] Many authors consider the "Shapovalov form" Φ_S on $\mathcal{V}$, which is slightly different. With respect to Φ_S the adjoint of $J^X_{(n)}$ is $J^{\omega(X)}_{(-n)}$, where ω is an anti-involution of $\mathfrak{g}$ (note this anti-involution is not $\mathfrak{g}$-invariant, so it could not have arisen from our construction, which so far has been fully $\mathfrak{g}$-invariant). The two are related by $\Phi_S(a,b) = \Phi(a, \omega b)$, where ω now is the corresponding involution of $\mathcal{V}$, which is self-adjoint for Φ.

(So this gives an alternate way of computing the chiral correlation system on $\mathbb{CP}^1$, and seeing that it is unique up to scale.)

Now we could also insert a third marked point, at z. Then the chiral correlation function gives $\Phi_{3,z} : \mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V} \to \mathbb{C}$. Its Laurent expansion in z is $\Phi_{3,z}(a,b,c) = \Phi(a, Y(b,z)c)$. In this sense Φ_3 is closely related to the vertex algebra operation. We would be tempted to say they are equivalent, if Φ is nondegenerate.

This raises the interesting question of whether Φ is actually nondegenerate or not. It is "easy" to show that the graded components of $\mathcal{V}$ are orthogonal under Φ. Also for $\mathfrak{g} = \mathfrak{gl}_1$ (Heisenberg) the partition basis is an orthogonal basis.

How about $\mathfrak{g} = \mathfrak{sl}_2$? Here we can fix a PBW basis and compute the determinant Δ_n of the matrix of Φ at each grade n. At the first few orders the result is:

$$\Delta_1 \propto k^3, \quad \Delta_2 \propto k^9(k-1)^5(k+2), \quad \Delta_3 \propto k^{22}(k-1)^{15}(k-2)^7(k+2)^4(3k+4)^3, \ldots$$

So there are some special values of k where Φ is degenerate: in these formulas we see already that this happens at $k = 1, 2, -2, -\frac{3}{4}$. The complete answer is that Φ is degenerate iff $k = -2 + p/q$ for some $p \geq 2$, $q > 0$, and $(p,q) = 1$. (So in particular, Φ is degenerate if $k \in \mathbb{Z}_{\geq 0}$.)

If Φ is degenerate, then we are in an interesting situation: it means there are some elements of $\mathcal{V}$ which "behave like zero" in all correlation functions on $\mathbb{CP}^1$. These elements form an ideal I in $\mathcal{V}$ (meaning I is stable under all $Y(a,z)$), in fact the unique maximal ideal. Then we can form a quotient $L_k(\mathfrak{g}) = \mathcal{V}/I$, which is a simple vertex algebra, i.e. it has no nontrivial ideals. If Φ is nondegenerate then $V^k(\mathfrak{g})$ is already simple.

Exercise 3.9. Prove that the ideal I is the unique maximal ideal in $\mathcal{V}$.

Exercise 3.10. Take $\mathfrak{g} = \mathfrak{sl}_2$ with the usual basis (E, F, H). Because $\Delta_2 = 0$ at $k = 1$, the ideal I must have a nontrivial grade 2 subspace. Prove that this is indeed so, by showing that the vector $J^E_{(-1)} J^E_{(-1)} |\Omega\rangle$ generates a nontrivial ideal. (Hint: show that all operators $J^X_{(n)}$ for $n > 0$ annihilate this vector.)

Exercise 3.11. Again take $\mathfrak{g} = \mathfrak{sl}_2$. Similarly to the previous exercise, find a vector at grade 2 which generates a nontrivial ideal at $k = -2$.

Exercise 3.12. Calculate the determinant Δ_2. (This seems tedious, but maybe fun to do once?)

Exercise 3.13. Suppose given any unpunctured curve C and any chiral correlation system for $V^k(\mathfrak{g})$. Then fix a point $p \in C$ and a coordinate system z around p, and consider the corresponding 1-point correlation function $\Phi : \mathcal{V} \to \mathbb{C}$. Letting p vary this gives a function $\Phi_z : \mathcal{V} \to \mathbb{C}$. Prove that $\partial_z \Phi_z(a) = \Phi_z(Ta)$.

3.3. **Including other modules.** Next we want to upgrade all this again to include more general $\mathcal{V}$-modules. So let us first quickly discuss some important examples of modules for $V^k(\mathfrak{g})$. Let $\widehat{\mathfrak{g}}$ be the Lie algebra generated by all modes $J^X_{(n)}$ (called *affine Kac-Moody algebra*). The representation theory of $V^k(\mathfrak{g})$ is closely related to that of $\widehat{\mathfrak{g}}$. For the latter the book [11] is very useful. Here we have to be brief.

Here is an important class of modules for $\mathcal{V} = V^k(\mathfrak{g})$.

Proposition 3.6. Suppose $\mathcal{R}$ is a *smooth* $\widehat{\mathfrak{g}}$-module: this means $\mathbf{1} \in \hat{\mathfrak{g}}$ acts as the identity and, for all χ, $J^X_{(n)}\chi = 0$ for $n \gg 0$. Then, $\mathcal{R}$ is also a $\mathcal{V}$-module.

(The action of $Y(J^X, z)$ comes directly from the $\hat{\mathfrak{g}}$-action; more general fields are determined by iterated contour-integration as usual.)

Example 3.7 (Parabolic Verma modules). Fix a $\mathfrak{g}$-module M. Let $\widehat{\mathfrak{g}}_{\geq 0}$ act on M as follows: the modes $X_{(0)}$ act through the $\mathfrak{g}$-action on M, and the positive modes $X_{(n)}$ act by 0. Then let $\mathcal{R} = \mathrm{Ind}^{\widehat{\mathfrak{g}}}_{\widehat{\mathfrak{g}}_{\geq 0}} M$.[6] It is graded, $\mathcal{R} = \bigoplus_{n\in\mathbb{N}} \mathcal{R}_n$, with $\mathcal{R}_0 = M$.

The case $M = \mathbf{triv}$ gives $\mathcal{R} = \mathcal{V}$ (called the *vacuum module*.)

Example 3.8 (Irregular modules). Fix an element $u \in \mathfrak{g}^*$. Then let $\widehat{\mathfrak{g}}_{\geq 1}$ act on $\mathbb{C}$ as follows: the modes $X_{(1)}$ act by $u(X)$, all higher modes $X_{(n)}$ act by 0. Then let $\mathcal{R} = \mathrm{Ind}^{\widehat{\mathfrak{g}}}_{\widehat{\mathfrak{g}}_{\geq 1}} \mathbb{C}$. This one is a little more exotic than the previous: in particular it is not graded, because it contains a vector $|\Omega_u\rangle$ with $X_{(1)}|\Omega_u\rangle = u(X)|\Omega_u\rangle$.

Example 3.9 (Hybrid modules). Again fix an element $u \in \mathfrak{g}^*$, let $\mathfrak{c} = Z(u)$ (the centralizer), and choose a $\mathfrak{c}$-module M. Then consider the subalgebra $\widehat{\mathfrak{g}}_{\geq 1} \oplus \widehat{\mathfrak{c}}_0$. This subalgebra acts on M as follows: the modes $X_{(0)}$ for $X \in \mathfrak{c}$ act through the $\mathfrak{c}$-action on M, the modes $X_{(1)}$ act by $u(X)\mathbf{1}_M$, all higher modes $X_{(n)}$ act by 0. (To check this is actually a Lie algebra action, we need to use the fact that $[\mathfrak{g}, \mathfrak{c}] \subset \ker u$.) Then let $\mathcal{R} = \mathrm{Ind}^{\widehat{\mathfrak{g}}}_{\widehat{\mathfrak{g}}_{\geq 1}+\widehat{\mathfrak{c}}_{\geq 0}} M$. This specializes to the previous two cases, upon choosing $u = 0$ or $M = \mathbf{triv}$. (Note that if u is a regular element then $\mathfrak{c} = 0$, so this generalization is not interesting in that case: we only get interesting hybrid modules when $0 \subsetneqq Z(u) \subsetneqq \mathfrak{g}$.)

Suppose we have $\mathcal{V}$-modules $\mathcal{R}_i$ and insert them at marked points $p_i \in C$, still with a local coordinate system z_i fixed around each p_i. Then:

Definition 3.10 (Chiral correlation system). A chiral correlation system for $V^k(\mathfrak{g})$ on C (where C now includes the marked points p_i, local coordinates z_i and modules $\mathcal{R}_i$) is a system of "functions"

$$\langle \prod_{r=1}^{k} J^{X_r}(q_r) \prod_{i=1}^{n} \cdot(p_i) \rangle : \left(\bigotimes_{i=1}^{n} \mathcal{R}_i \right) \to \left(\otimes_{i=1}^{k} T^*_{q_i} C \right)$$

obeying all the conditions in Definition 3.1, plus a Laurent-expansion condition as $q_i \to p_j$,

$$\langle \cdots J^X(q)\chi(p) \rangle = \sum_{n\in\mathbb{Z}} z(q)^{-n-1} \langle \cdots (J^{X,\mathcal{R}}_{(n)}\chi)(p) \rangle \,.$$

Note that, in contrast to the case where all $\mathcal{R}_i = \mathcal{V}$, here having chiral correlation system without marked points does *not* automatically construct for us a chiral correlation system with marked points.

Example 3.11. Suppose again that $C = \mathbb{CP}^1$ and we insert $\mathcal{V}$-modules $\mathcal{R}$, $\mathcal{R}'$ at 0 and ∞, with local coordinates as before.

[6] Recall the definition: for any $\mathfrak{h} \subset \mathfrak{g}$ (or more generally any map of Lie algebras $\mathfrak{h} \to \mathfrak{g}$), and any $\mathfrak{h}$-module M, we get a $\mathfrak{g}$-module $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{h}} M = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} M$.

Then, by the same method as before, a chiral correlation system on C gives in particular a pairing $\Phi : \mathcal{R} \otimes \mathcal{R}' \to \mathbb{C}$ obeying again the condition $\Phi(J^X_{(n)}\chi, \psi) = -\Phi(\chi, J^X_{(-n)}\psi)$. Such a pairing might or might not exist, depending on what $\mathcal{R}$ and $\mathcal{R}'$ are.

For instance, suppose $\mathcal{R} = \mathrm{Ind}^{\widehat{\mathfrak{g}}}_{\mathfrak{g}_{\geq 0}} M$ as in Example 3.7, and similarly $\mathcal{R}'$. Then we can restrict attention to the zero-grade parts M, M', which are closed under the action of the zero-modes $J^X_{(0)}$. The invariance condition $\Phi(J^X_{(0)}\chi, \psi) = -\Phi(\chi, J^X_{(0)}\psi)$ just says that Φ gives a $\mathfrak{g}$-invariant pairing $M \otimes M' \to \mathbb{C}$. Moreover, similarly to the case we discussed above, the full Φ is determined (and not overdetermined) by its restriction to $M \otimes M'$, and once we have Φ we can sum Laurent series to build a chiral correlation system. The conclusion is that chiral correlation systems on C correspond to $\mathfrak{g}$-invariant pairings $M \otimes M' \to \mathbb{C}$. In particular, if M and M' are irreducible, then the space of chiral correlation systems has dimension 1 if $M' \simeq M^*$, and dimension 0 otherwise.

Another interesting example: suppose $\mathcal{R} = \mathrm{Ind}^{\widehat{\mathfrak{g}}}_{\mathfrak{g}_{\geq 0}} M$ as above, but $\mathcal{R}'$ is an irregular module as in Example 3.8. In this case by a similar analysis we conclude that a chiral correlation system is uniquely determined by the values $\Phi(m \otimes |\Omega_u\rangle)$ for $m \in M$, and these values are unconstrained. Thus the space of chiral correlation systems as a vector space is M^*.

Example 3.12. Now suppose that again that $C = \mathbb{CP}^1$ and we insert arbitrary $\mathcal{V}$-modules $\mathcal{R}_i$ at marked points p_i, with arbitrary local coordinates z_i.

If we have a chiral correlation system, then as before, we get a map

$$\Phi : \bigotimes_{i=1}^n \mathcal{R}_i \to \mathbb{C}\,.$$

This map is subject to many conditions. To get the simplest of these conditions we consider the 1-form $\eta(z) = \langle J^X(z) \prod_{i=1}^n \chi_i(p_i)\rangle$. This 1-form is meromorphic, with singularities only at the p_i, on the compact Riemann surface C, so the sum of its residues must be zero. That gives the condition

$$\sum_{i=1}^n \Phi(\chi_1, \ldots, J^{X,\mathcal{R}_i}_{(0)}\chi_i, \ldots, \chi_n) = 0\,. \tag{3.2}$$

More generally we can consider the 1-form $f(z)\eta(z)$, where f is any meromorphic function on C which is holomorphic away from the p_i. Then we get a more general condition

$$\sum_{i=1}^n \mathrm{Res}_{p_i}(f\eta) = 0\,.$$

So what are these residues? Laurent expand near each p_i as

$$f = \sum_{m\in\mathbb{Z}} f_{i,m}\, z_i^m\,.$$

Then the condition is

$$\sum_{i=1}^n \sum_{m\in\mathbb{Z}} f_{i,m}\Phi(\chi_1, \ldots, J^{X,\mathcal{R}_i}_{(m)}\chi_i, \ldots, \chi_n) = 0\,. \tag{3.3}$$

Conversely, given Φ which obeys (3.3) for every f, we can build a chiral correlation system. (This is a nontrivial claim: it says that the Laurent series we build out of Φ actually converge! [...])

Another way to phrase the condition (3.3): there is a linear subspace $K \subset W = \otimes_i \mathcal{R}_i$ consisting of vectors of the form

$$\sum_{i=1}^{n} \sum_{m \in \mathbb{Z}} \chi_1 \otimes \cdots \otimes (f_{i,m} J^{X,\mathcal{R}_i}_{(m)}) \chi_i \otimes \cdots \otimes \chi_n$$

and Φ must annihilate K.

Now how big is K? Let's consider the case where $\mathcal{R}_i = \mathrm{Ind}^{\widehat{\mathfrak{g}}}_{\widehat{\mathfrak{g}}_{\geq 0}} M_i$. First note that for any i and any $m < 0$, the function $f(z) = (z - z_i)^m$ has $f_{i,m} = 1$ and all other $f_{j,m} = 0$ for $m < 0$. It follows that $K \oplus W_0 = W$. Next note that any element in $S \cap W_0$ must come from a constant function f (since these are the only functions on $\mathbb{CP}^1$ which has $f_{i,m} = 0$ for all $m < 0$). It follows that any $\Phi_0 : W_0 \to \mathbb{C}$ which obeys the condition (3.2) has a unique extension to $\Phi : W \to \mathbb{C}$ obeying (3.3). Said otherwise, the space of chiral correlation systems on C is

$$\mathrm{Hom}^{\mathfrak{g}}(\bigotimes_{i=1}^{n} M_i, \mathbf{triv}) \,.$$

Our last discussion motivates the following:

Definition 3.13 (Conformal block for affine algebra). Suppose C is a Riemann surface with marked points p_i, local coordinates z_i, and $V^k(\mathfrak{g})$-modules $\mathcal{R}_i$. A *conformal block* for $V^k(\mathfrak{g})$ on C is a linear map $\Phi : \bigotimes_i \mathcal{R}_i \to \mathbb{C}$ which annihilates all vectors of the form

$$\sum_{i=1}^{n} \sum_{m \in \mathbb{Z}} \chi_1 \otimes \cdots \otimes (f_{i,m} J^{X,\mathcal{R}_i}_{(m)}) \chi_i \otimes \cdots \otimes \chi_n$$

where f is any global meromorphic function on C, with poles only at p_i.

Proposition 3.14 (Conformal blocks vs. chiral correlation systems). Fix C with markings as above. By Fourier expansion we obtain a linear map from the space of chiral correlation systems for $V^k(\mathfrak{g})$ on C to the space of conformal blocks. This map is injective if C is connected and has at least one marked point.

For a completely arbitrary conformal block it is not clear that summing up the formal Laurent series will give an actual function. As a piece of personal philosophy: the *most interesting* conformal blocks are the ones which actually come from chiral correlation systems.

3.4. **Changes of coordinate, by hand.** Given a chiral correlation system for $V^k(\mathfrak{g})$ and any choice of marked points with local coordinate systems, we found a corresponding conformal block $\Phi : \otimes_i \mathcal{V} \to \mathbb{C}$. Now we can ask: what is the relation between these blocks for different choices of marked points and coordinates? We already know the answer, in a sense: we built into our definition of chiral correlation systems that the J^X are global 1-forms; and all the other elements of $\mathcal{V}$ are built out of J^X; so their transformation laws are all in principle determined. But let us try to spell this out a bit more.

Let $\mathrm{Aut}\,\mathbb{C}[[t]]$ denote the group of changes of coordinates on the formal disc. The elements of $\mathrm{Aut}\,\mathbb{C}[[t]]$ are formal power series $a_1 t + a_2 t^2 + \cdots$ with $a_1 \neq 0$, which multiply by formal composition.

Definition 3.15. We say $\mathrm{Aut}\,\mathbb{C}[[t]]$ acts on a vertex algebra $\mathcal{V}$ if we have $\rho : \mathrm{Aut}\,\mathbb{C}[[t]] \to \mathrm{End}\,\mathcal{V}$ obeying

$$\rho(g) Y(a,z) \rho(g)^{-1} = Y(\rho(g_z)a, g(z)) \,. \tag{3.4}$$

Here $g_z \in \operatorname{Aut}\mathbb{C}[[t]]$ is the pointed version of g at z, defined by $g_z(t) = g(z+t) - g(z)$.

To construct such a $\operatorname{Aut}\mathbb{C}[[t]]$ action we can start with the infinitesimal version, i.e. consider the *Witt algebra*, the Lie algebra $\operatorname{Der}\mathbb{C}[[t]]$ of vector fields on the formal disc which vanish at the origin. If ℓ_n denotes the vector field $-t^{n+1}\partial_t$ then we have for all $n, m \geq 0$

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m}\,. \tag{3.5}$$

Now, to construct a $\operatorname{Der}\mathbb{C}[[t]]$ action on $\mathcal{V} = V^k(\mathfrak{g})$, write

$$J^X_{(m)} = \oint z^m J^X(z)$$

and then consider the effect of $z^{n+1}\partial_z$: it acts only on the explicit factor z^m, giving

$$[\ell_n, J^X_{(m)}] = -\oint z^{n+1}\partial_z z^m J^X(z) = -mJ^X_{(m+n)}\,. \tag{3.6}$$

Combining this with $\ell_n|\Omega\rangle = 0$ determines an action of $\operatorname{Der}\mathbb{C}[[t]]$ on $\mathcal{V}$, which can be integrated to an action of $\operatorname{Aut}\mathbb{C}[[t]]$.

Exercise 3.14. Prove that this action of $\operatorname{Aut}\mathbb{C}[[t]]$ on $\mathcal{V}$ obeys (3.4). [check, give an appropriate hint!]

An element $a \in \mathcal{V}$ which is annihilated by all positive modes ℓ_n, and obeys $\ell_0 a = ha$, is called a *primary field* of *conformal weight* h. For example, the elements $J^X \in \mathcal{V}$ are primary fields of conformal weight 1. The transformation law of primary fields under changes of coordinate is relatively simple: if $g(t) = c_1 t + c_2 t^2 + \cdots$ then $\rho(g)a = (c_1)^h a$. Said otherwise, a "transforms as an h-differential." So if $h = 1$ then a transforms as a 1-form (this is the case for $a = J^X$), if $h = 2$ then a transforms as a quadratic differential, and so on.

It follows directly from our iterated contour-integral construction that the conformal blocks we construct are compatible with the $\operatorname{Aut}\mathbb{C}[[t]]$-action:

Proposition 3.16 (Coordinate covariance of conformal blocks, with the vacuum module). Fix a chiral correlation system on a curve C. For any choice of local coordinates z_i at marked points p_i, let

$$\Phi_{(z_i)} : \bigotimes_{i=1}^{n} \mathcal{V} \to \mathbb{C}$$

be the conformal block obtained from the correlation system. If we replace the local coordinates z_i by

$$\tilde{z}_i = g_i(z_i) \qquad (i = 1, \ldots, n),$$

then the corresponding block is given by

$$\Phi_{(\tilde{z}_i)} = \Phi_{(z_i)} \circ \big(\rho(g_1) \otimes \cdots \otimes \rho(g_n)\big)\,.$$

In a similar fashion we can define compatibility for a $\operatorname{Aut}\mathbb{C}[[t]]$-action on a $\mathcal{R}$-module: this means we require $\rho^{\mathcal{R}} : \operatorname{Aut}\mathbb{C}[[t]] \to \operatorname{End}\mathcal{R}$ with

$$\rho^{\mathcal{R}}(g) Y^{\mathcal{R}}(a, z) \rho^{\mathcal{R}}(g)^{-1} = Y^{\mathcal{R}}(\rho(g_z)a, g(z))\,.$$

Proposition 3.17 (Coordinate covariance of conformal blocks, with more general modules). Suppose $\mathcal{R}_i$ are $\mathcal{V}$-modules, with a compatible $\operatorname{Aut}\mathbb{C}[[t]]$-action. Suppose C is a Riemann surface with marked points p_i and local coordinates z_i, and

$$\Phi : \bigotimes_{i=1}^{n} \mathcal{R}_i \to \mathbb{C}$$

is a conformal block. If we define

$$\tilde{z}_i = g_i(z_i) \qquad (i = 1, \dots, n),$$

then

$$\Phi_{(\tilde{z}_i)} = \Phi_{(z_i)} \circ \big(\rho(g_1) \otimes \cdots \otimes \rho(g_n)\big)$$

is also a conformal block, now with the local coordinates $\tilde{z}_i$.

3.5. **Chiral correlation systems and conformal blocks for vertex algebras with $\operatorname{Aut}\mathbb{C}[[t]]$-action.** Using this we can finally define chiral correlation systems and conformal blocks, now for any vertex algebra with a compatible $\operatorname{Aut}\mathbb{C}[[t]]$-action, instead of just $\mathcal{V} = V^k(\mathfrak{g})$. There are variants depending on how we treat the coordinate systems around the marked points.

Definition 3.18 (Torsor of formal local coordinates). Fix $p \in C$. Let Coord_p be the set of formal local coordinate systems z around p, with $z(p) = 0$.

Coord_p is a torsor for $\operatorname{Aut}\mathbb{C}[[t]]$. Moreover, the Coord_p fit together into a principal $\operatorname{Aut}\mathbb{C}[[t]]$-bundle Coord_C over C. There are distinguished local trivializations given as follows: fix a local coordinate z on a patch U, then assign to $p \in U$ the local coordinate $t_p = z - z(p)$, which has $t_p(p) = 0$ as needed.

Definition 3.19 (Twisting of $\mathcal{V}$-modules by local coordinate systems). Fix $\mathcal{R}$ a $\mathcal{V}$-module with compatible $\operatorname{Aut}\mathbb{C}[[t]]$-action. Then define an associated (infinite rank) vector bundle

$$\mathcal{R}_C = \operatorname{Coord}_C \times_{\operatorname{Aut}\mathbb{C}[[t]]} \mathcal{R}\,.$$

(In particular we can apply this when $\mathcal{R} = \mathcal{V}$.)

Each fiber $\mathcal{R}_p$ can be identified with $\mathcal{R}$, but not canonically: one gets an identification only after choosing an element in the fiber Coord_p, i.e., a formal local coordinate at p.

Example 3.20 (Basic sections of the bundle of vertex algebras). Suppose $C = \mathbb{C}$. Then Coord_C can be globally trivialized over $\mathbb{C}$ by the standard local coordinate z. This trivializes the bundle $\mathcal{V}_C$, so any holomorphic $\mathcal{V}$-valued function $a(z)$ gives a section; in particular any element $a \in \mathcal{V}$ gives a section.

Example 3.21 (A change of coordinates in the bundle of vertex algebras). Suppose $C = \mathbb{CP}^1$ and $a \in \mathcal{V}$. Then Coord_C can be globally trivialized over the south-pole patch by the standard local coordinate z, and over the north-pole patch by $\zeta = z^{-1}$. The element a induces a section of $\mathcal{V}_C$ over the south-pole patch. For p in the overlap, write $t_S = z - z(p)$, $t_N = \zeta - \zeta(p)$. The pointed change of coordinate from t_N to t_S at p is

$$\gamma_p(t) = \frac{1}{\zeta(p) + t} - \frac{1}{\zeta(p)} \in \operatorname{Aut}\mathbb{C}[[t]].$$

In the north-pole trivialization, the section is represented by $\rho(\gamma_p)a$. In particular, if a is primary of weight h, then this action is determined by the 1-jet $\mathrm{d}\gamma_p(t)/\mathrm{d}t|_{t=0} = -\zeta(p)^{-2}$, giving

$$\rho(\gamma_p)a = \left(-\zeta(p)^{-2}\right)^h a.$$

Now suppose $\mathcal{R}$ is graded. In this case we can consider the filtered components $\mathcal{R}_{\leq n}$. Each of these is also acted on by $\mathrm{Aut}\,\mathbb{C}[[t]]$[7] so we have a corresponding filtration of $\mathcal{R}_p$ by $\mathcal{R}_{p,\leq n}$. For any n, the $\mathcal{R}_{p,\leq n}$ are the fibers of a vector bundle $\mathcal{R}_{C,\leq n}$ over C. So the infinite rank bundle $\mathcal{R}_C$ is filtered by the finite rank bundles $\mathcal{R}_{C,\leq n}$.

We can use these constructions to define chiral correlation systems in a way that is independent of the local coordinate:

Definition 3.22 (Chiral correlation systems for a vertex algebra and modules with $\mathrm{Aut}\,\mathbb{C}[[t]]$-action). Suppose $\mathcal{V}$ is a vertex algebra with a compatible $\mathrm{Aut}\,\mathbb{C}[[t]]$-action, and C is a Riemann surface, with marked points p_i, and $\mathcal{V}$-modules $\mathcal{R}_i$ with compatible $\mathrm{Aut}\,\mathbb{C}[[t]]$-actions. Then a *chiral correlation system* is a collection

$$\langle \prod_{i=1}^{n} a_i(q_i) \prod_{i=1}^{k} \chi_i(p_i) \rangle \in \mathbb{C}$$

where each $a_i \in \mathcal{V}_{q_i}$, $\chi_i \in \mathcal{R}_{p_i}$, such that:

- The $\langle \cdots \rangle$ are symmetric in the q_i.
- The $\langle \cdots \rangle$ are multilinear in the a_i and χ_i.
- The $\langle \cdots \rangle$ vary meromorphically with each q_i.
- Fix s either p_i or q_i, and choose a local coordinate z around s. We describe the singular part of $\langle \cdots \rangle$ as some $q_j \to s$. Let $\mathcal{R}$ be the $\mathcal{V}$-module corresponding to s. Given $a \in \mathcal{V}$, and q near s, use the local coordinate z to produce $a(q) \in \mathcal{R}_q$. Then, for any $\chi \in \mathcal{R}$, we have
$$\langle \cdots a(q)\chi(s) \rangle = \sum_{n \in \mathbb{Z}} z(q)^{-n-1} \langle \cdots (a_{(n)}\chi)(p) \rangle .$$

Likewise we can define conformal blocks in a way that is independent of the local coordinate:

Definition 3.23. Suppose $\mathcal{V}$ is a vertex algebra with $\mathrm{Aut}\,\mathbb{C}[[t]]$-action. Suppose C is a Riemann surface with marked points p_i, and $\mathcal{V}$-modules $\mathcal{R}_i$ with compatible $\mathrm{Aut}\,\mathbb{C}[[t]]$-action. Let $\mathcal{V}(C)$ denote the space of global meromorphic sections of $\mathcal{V}_C \otimes K_C$, with poles only at the p_i. Fix a local coordinate z_i around each p_i, and for any $a \in \mathcal{V}(C)$ write its local expansion around p_i as $\sum_{m \in \mathbb{Z}} a_{i,m} z_i^m \mathrm{d}z_i$, with each $a_{i,m} \in \mathcal{V}$. Then, for any $a \in \mathcal{V}(C)$, define

$$o(a) \in \mathrm{End} \bigotimes_i (\mathcal{R}_i)_{p_i}$$

given by

$$o(a) = \sum_{i=1}^{n} \sum_{m \in \mathbb{Z}} \mathbf{1} \otimes \cdots \otimes (a_{i,m})_{(m)} \otimes \cdots \otimes \mathbf{1}$$

where we use the local coordinate z_i to identify $\mathcal{R}_i$ with $(\mathcal{R}_i)_{p_i}$.

[7] The action of $\mathrm{Aut}\,\mathbb{C}[[t]]$ does not preserve the grading, but does preserve this filtration.

We could alternatively write this

$$o(a) = \sum_{i=1}^{n} \mathbf{1} \otimes \cdots \otimes \operatorname{Res}_{p_i} a \otimes \cdots \otimes \mathbf{1}$$

Definition 3.24 (Conformal blocks for a vertex algebra and modules with $\operatorname{Aut}\mathbb{C}[[t]]$-action). Define the subspace

$$K(C) = \operatorname{Span}\{\operatorname{Im} o(a) \mid a \in \mathcal{V}(C)\} \tag{3.7}$$

and

$$\operatorname{Coinv}(C) = \left(\bigotimes_i (\mathcal{R}_i)_{p_i}\right) / K(C)\,.$$

A *conformal block* for $\mathcal{V}$ on C is a linear map

$$\Phi : \operatorname{Coinv}(C) \to \mathbb{C}\,.$$

In the case $\mathcal{V} = V^k(\mathfrak{g})$ this recovers Definition 3.13. (To compare the two, note that any meromorphic function f on C induces a meromorphic section fJ^X of $\mathcal{V}_C \otimes K_C$.)

Exercise 3.15. Prove that the subspace $K(C)$ is independent of the local coordinate z_i we use around p_i. (Hint: if $\tilde{z}_i = g_i(z_i)$ then the identification of $(\mathcal{R}_i)_{p_i}$ with $\mathcal{R}_i$ changes by $\rho(g_i)$. Check that the local mode expansions are conjugated accordingly, so the span of the $\operatorname{Im} o(a)$ is unchanged.)

Exercise 3.16. Prove that a chiral correlation system in the sense of Definition 3.22 induces a conformal block in the sense of Definition 3.24. (Hint: as usual, use iterated Laurent expansions of the correlation functions to define Φ.)

There is also a variant of all the above where we consider only an action of the subgroup $\operatorname{Der}\mathbb{C}[[t]]_+$, consisting of power series with $a_1 = 1$, i.e. $z + a_2 z^2 + a_3 z^3 + \cdots$ In this case we cannot completely eliminate the coordinate dependence, but we can reduce it to a dependence on the 1-jet of the coordinate, or dually a nonzero tangent vector at the marked point.

3.6. **Changes of coordinate, by Virasoro.** Now let us reconsider the $\operatorname{Aut}\mathbb{C}[[t]]$-action on $V^k(\mathfrak{g})$ from a different point of view. We specialize very slightly to the case $k \neq -h^\vee$. The idea is to find a copy of $\operatorname{Der}\mathbb{C}[[t]]$ inside $\mathcal{V}$, and then use the general fact that $\mathcal{V}$ acts on itself. For this consider $\omega \in \mathcal{V}$ defined by

$$\omega = \frac{1}{2(k+h^\vee)} \sum_i J^{X^i}_{(-1)} J^{X_i}_{(-1)} |\Omega\rangle$$

where X^i, X_i are dual bases of $\mathfrak{g}$ with respect to κ, and $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ (see Appendix A). The key fact about ω is:

Proposition 3.25 (Sugawara construction). The λ-bracket of ω with itself and with J^X are

$$[\omega_\lambda \omega] = \frac{c}{12}\lambda^3 \mathbf{1} + (2\lambda + \partial)\omega, \qquad [\omega_\lambda J^X] = (\partial + \lambda) J^X\,. \tag{3.8}$$

with

$$c = \frac{k \dim \mathfrak{g}}{k + h^\vee}\,.$$

In principle one could prove this directly by hand, carefully manipulating the singular parts of the OPEs to figure out what singularity we get when we bring $\omega(z)$ close to $\omega(w)$. But it seems a bit hairy to do so. It might be better to introduce the "noncommutative Wick formula"[8]

Lemma 3.26 (Noncommutative Wick formula).

$$[a_\lambda : bc :] = : [a_\lambda b]\, c : + : b\, [a_\lambda c] : + \int_0^\lambda \Big[[a_\lambda b]_\mu\, c \Big]\, \mathrm{d}\mu\,. \tag{3.9}$$

This lemma is an efficient way of keeping track of the various singular terms which appear when we bring composite objects together. To get an idea of the proof of the lemma we can use successive contour-integration: indeed the two operations $: ab :$ and $[a_\lambda b]$ are both defined by contour-integrals. The left side of (3.9) can be written as [make figure with contours]

$$\oint_x \mathrm{d}z\, \mathrm{e}^{\lambda(z-x)} a(z) \left(\oint_x \frac{\mathrm{d}w}{w-x} b(w) c(x) \right)$$

By deformation of the z contour this turns into

$$\oint_x \frac{\mathrm{d}w}{w-x} \left(\oint_w \mathrm{d}z\, \mathrm{e}^{\lambda(z-x)} a(z) b(w) \right) c(x) + \oint_x \frac{\mathrm{d}w}{w-x} \left(\oint_x \mathrm{d}z\, \mathrm{e}^{\lambda(z-x)} a(z) b(w) \right) c(x)\,.$$

The second term is $: b\, [a_\lambda c] :$, while the first differs from $: [a_\lambda b]\, c :$ by the error term

$$\oint_x \frac{\mathrm{d}w}{w-x} \left(\oint_w \mathrm{d}z\, [\mathrm{e}^{\lambda(z-x)} - \mathrm{e}^{\lambda(z-w)}] a(z) b(w) \right) c(x)\,. \tag{3.10}$$

We can re-expand the function appearing here as

$$\frac{\mathrm{e}^{\lambda(z-x)} - \mathrm{e}^{\lambda(z-w)}}{w-x} = \mathrm{e}^{\lambda(z-w)} \int_0^\lambda \mathrm{d}\mu\, \mathrm{e}^{\mu(w-x)}$$

and thus recognize the error term (3.10) as

$$\int_0^\lambda \mathrm{d}\mu \left[\oint_x \mathrm{d}w\, \mathrm{e}^{\mu(w-x)} \left(\oint_w \mathrm{d}z\, \mathrm{e}^{\lambda(z-w)} a(z) b(w) \right) c(x) \right]$$

which is

$$\int_0^\lambda \mathrm{d}\mu\, \Big[[a_\lambda b]_\mu\, c \Big]$$

as desired.

Now to prove (3.8) we work in a κ-orthonormal basis (X^i) for $\mathfrak{g}$, use J^i for J^{X^i}, and write $[X^i, X^j] = f^{ijk} X^k$, with the convention that we always sum over repeated indices. Then (3.9) gives

$$\big[J^i_\lambda : J^j J^j : \big] = : \big[J^i_\lambda J^j \big]\, J^j : + : J^j \big[J^i_\lambda J^j \big] : + \int_0^\lambda \Big[\big[J^i_\lambda J^j \big]_\mu\, J^j \Big]\, \mathrm{d}\mu\,,$$

and recalling $\big[J^k_\lambda J^\ell \big] = \lambda k \delta^{k\ell} \mathbf{1} + f^{k\ell m} J^m$ this is

$$= : (\lambda k \delta^{ij} \mathbf{1} + f^{ijm} J^m) J^j : + : J^j (\lambda k \delta^{ij} \mathbf{1} + f^{ijm} J^m) : + \int_0^\lambda \big[(\lambda k \delta^{ij} \mathbf{1} + f^{ijm} J^m)_\mu J^j \big]\, \mathrm{d}\mu\,,$$

[8] I had trouble chasing down a good reference for this formula; [7] seems to give a version for the mode expansions; [12] gives the formula as we wrote it here. Warning: [13] has a slightly different formula, hopefully a typo there.

which simplifies to

$$= 2\lambda k J^i + f^{ijm}(:J^m J^j: + :J^j J^m:) + \int_0^\lambda f^{ijm} f^{mj\ell} J^\ell \, \mathrm{d}\mu \,.$$

Now using the asymmetry of f^{ijm} and $f^{ijm} f^{mj\ell} = 2h^\vee \delta^i_\ell$ (see Appendix A), this simplifies to

$$\left[J^i_\lambda : J^j J^j :\right] = 2(k+h^\vee)\lambda J^i$$

and using the skew-symmetry property of λ-brackets this is also

$$\left[: J^j J^j :_\lambda J^i\right] = 2(k+h^\vee)(\lambda+\partial) J^i \,.$$

Then dividing out the factor $2(k+h^\vee)$ we get the second equation in (3.8). Next we want to compute $[\omega_\lambda \omega]$; for that we use (3.9) again to reduce $[: J^i J^i :_\lambda : J^j J^j :]$ to things we have already computed.

Exercise 3.17. Finish the proof of (3.8).

The equations (3.8) are extremely important. The first equation in (3.8) is the λ-bracket of the generator of the Virasoro algebra. It implies that there is a homomorphism of vertex algebras, $\mathrm{Vir}^c \to V^k(\mathfrak{g})$. Also if we write

$$Y(\omega, z) = \sum L_n z^{-n-2}$$

(note the shifted index: $L_n = \omega_{(n+1)}$) then we have

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m+1)(m)(m-1)\,\delta_{m+n,0}\mathbf{1} \,,$$

which is the Virasoro Lie algebra. (3.8) also implies

$$[L_n, J^X_m] = -m J^X_{m+n} \,. \tag{3.11}$$

Example 3.27. If $\mathfrak{g} = \mathfrak{sl}_2$ then $h^\vee = 2$, so $c = \frac{3k}{k+2}$.

Example 3.28. If $\mathfrak{g} = \mathfrak{gl}_1$ then $h^\vee = 0$, so $c = 1$, independent of k.

Now if we consider only the modes L_n with $n \geq 0$, the central charge c does not appear, and we just see the Witt algebra (3.5). Thus this is a second way of getting an action of the Witt algebra on $\mathcal{V}$. Fortunately it actually coincides with the first way, because the brackets (3.11) agree with (3.6).

3.7. **Consequences of Virasoro.** Moreover, once we realize the action of the Witt algebra "internally" via the element $\omega \in \mathcal{V}$, it automatically acts not only on $\mathcal{V}$ but on all $\mathcal{V}$-modules $\mathcal{R}$. Now we should ask whether it integrates from $\mathrm{Der}\,\mathbb{C}[[t]]$ to the group $\mathrm{Aut}\,\mathbb{C}[[t]]$ of coordinate changes. The answer depends on $\mathcal{R}$.

The best possible situation arises if L_0 acts diagonalizably on $\mathcal{R}$, with all eigenvalues integers and bounded below. In that case L_0 provides a grading on $\mathcal{R}$. Moreover $\mathrm{Der}\,\mathbb{C}[[t]]_+$ acts locally nilpotently[9] and $\mathrm{Der}\,\mathbb{C}[[t]]_0$ acts diagonally respecting the relation $\exp(2\pi \mathrm{i} L_0) = 1$. Thus both indeed integrate to $\rho : \mathrm{Aut}\,\mathbb{C}[[t]] \to \mathrm{End}\,\mathcal{R}$.

The next best situation arises if $\mathcal{R}$ is still graded by L_0 but the eigenvalues are not necessarily integers. Then at least $\mathrm{Der}\,\mathbb{C}[[t]]_+$ integrates to $\rho : \mathrm{Aut}\,\mathbb{C}[[t]]_+ \to \mathrm{End}\,\mathcal{R}$.

It is interesting to consider the transformation property of the Virasoro field $\omega = L_{-2}|\Omega\rangle$ itself. This one has $L_0\omega = 2\omega$, so you might think it would transform as a quadratic differential, but it is

[9] *Locally nilpotently* means that for every $v \in \mathrm{Der}\,\mathbb{C}[[t]]_+$ and $\chi \in \mathcal{R}$ there exists N such that $v^N \chi = 0$.

not primary for $c \neq 0$: indeed $L_2\omega = [L_2, L_{-2}]|\Omega\rangle = \frac{c}{2}|\Omega\rangle$. Thus its transformation law is more interesting: under a change of coordinates $g(z) = a_1 z + a_2 z^2 + \cdots$ we have

$$\begin{aligned} \rho(g)\omega &= a_1^2\,\omega + \frac{c}{2}\,\frac{a_1 a_3 - a_2^2}{a_1^2}|\Omega\rangle \\ &= \left((\mathrm{d}g/\mathrm{d}z)^2\omega + \frac{c}{12}\{g,z\}|\Omega\rangle\right)|_{z=0}\,. \end{aligned} \tag{3.12}$$

where $\{\cdot,\cdot\}$ denotes the Schwarzian derivative: $\{g(z), z\} = \frac{g'''(z)}{g'(z)} - \frac{3}{2}\left(\frac{g''(z)}{g'(z)}\right)^2$.

Exercise 3.18. Prove (3.12), by (in an appropriate sense) integrating the Virasoro algebra.

The Schwarzian derivative is a sort of unfamiliar object to many of us. Nevertheless it appears here so we have to grapple with it a little. One useful fact about it is that $\{g(z), z\} = 0$ if and only if g is a Möbius transformation. Here is another way to think about this: the group $\mathrm{PSL}(2,\mathbb{C})$ of Möbius transformations acting on $\mathbb{CP}^1$ is generated by $\ell_{-1} = -\partial_z$, $\ell_0 = -z\partial_z$, $\ell_1 = -z^2\partial_z$. (Indeed these three span the space of holomorphic vector fields on $\mathbb{CP}^1$.) Thus, for a Möbius transformation $g(z)$, the computation of $\rho(g)\omega$ cannot involve the central charge c — that can only appear if we use ℓ_2. This matches the fact that for a Möbius transformation we have $\{g(z), z\} = 0$.

A few other useful properties of the Schwarzian derivative:

Proposition 3.29. We have:

(1) $\{M \circ f(z), z\} = \{f(z), z\}$ if M is a Möbius transformation.

(2) $\{f \circ g, z\} = \{f \circ g, g\}(g'(z))^2 + \{g, z\}$.

3.8. **Moving the marked points.** So far we have only used the L_n with $n \geq 0$. Another important feature is that L_{-1} acting on $\mathcal{V}$ generates translations:

$$L_{-1} = T\,.$$

(To prove this it's enough to use the fact $[L_{-1}, J^X_{(m)}] = -mJ^X_{(m-1)}$.) This means in particular that chiral correlation functions obey the relation

$$\partial_z\langle a(z)\cdots\rangle = \langle (L_{-1}a)(z)\cdots\rangle$$

One would like to integrate this statement, i.e. to write:

$$\langle a(z+w)\cdots\rangle = \langle (\mathrm{e}^{wL_{-1}}a)(z)\cdots\rangle$$

This equation is indeed true, in the sense that the expansion of the right side in powers of w converges to the left side, so long as $|w|$ is smaller than the distance from z to the nearest other insertion.

What is more interesting is that we can do the same thing with $\mathcal{V}$-modules inserted, i.e. for $\chi \in \mathcal{R}$ we can have the equation

$$\partial_z\langle \chi(z)\cdots\rangle = \langle (L_{-1}\chi)(z)\cdots\rangle$$

To formulate this precisely we work in the language of conformal blocks rather than chiral correlation systems. Fix a base curve C with punctures and modules, and let one puncture p vary while the others p_i are held fixed. Then the spaces $\mathcal{R}_p \otimes \bigotimes_i (\mathcal{R}_i)_{p_i}$ are the fibers of an infinite rank vector bundle $\mathcal{B}$ over C, with a subsheaf $\mathcal{B}_K$ given by the spaces $K(C_p)$ from (3.7).

Now, suppose we fix a local coordinate system z, and let p be a moving marked point in the z-patch. Then:

Proposition 3.30. If $s(p)$ is a section of $\mathcal{B}_K$, then $(\partial_{z(p)} + L_{-1})s(p)$ is also a section of $\mathcal{B}_K$. (Here L_{-1} acts in the tensor factor $\mathcal{R}_p$ of $\mathcal{B}$.)

Example 3.31. Suppose $\mathcal{V} = \mathrm{Heis}$ and $C = \mathbb{CP}^1$ with marked points at w, ∞. (So in the above notation we are taking $w = z(p)$.) Then consider the global section zJ of $\mathcal{V}(C)$, with (using $z = (z-w) + w = (z^{-1})^{-1}$)

$$o(zJ) = (J_1 + wJ_0) \otimes \mathbf{1} + \mathbf{1} \otimes J_{-1}\,.$$

Then using $[L_{-1}, J_1] = -J_0$ and $[L_{-1}, J_0] = 0$ we see that

$$[\partial_w + L_{-1} \otimes \mathbf{1}, o(zJ)] = 0\,.$$

It follows that if s is a section of $\mathcal{B}_K$ of the form $s(w) = o(zJ)\chi$, then indeed $(\partial_w + L_{-1} \otimes \mathbf{1})s(w) = o(zJ)(\partial_w + L_{-1})\chi$ which is also a section of $\mathcal{B}_K$, as desired.

More generally, here is the proof. As in the example, let z denote a fixed local coordinate, and let p be a moving marked point in the z-coordinate patch, with $w = z(p)$. We consider a section $a(w)$ of the vertex algebra bundle $\mathcal{V}(C_w)$, whose expansion in the z-patch is

$$a(w) = \sum_m a_m(w)(z-w)^m\,.$$

Then we have

$$\partial_w a_m(w) = (m+1)a_{m+1}(w) + (\partial_w a)_m(w)$$

and thus

$$\partial_w(o(a)) = \sum_m (m+1)(a_{m+1})_{(m)} + o(\partial_w a)\,.$$

On the other hand, we have

$$[L_{-1}, o(a)] = \sum_m [L_{-1}, (a_m)_{(m)}] = -\sum_m m(a_m)_{(m-1)}$$

and combining these two we get

$$[\partial_w + L_{-1}, o(a)] = o(\partial_w a)\,. \tag{3.13}$$

(In the example we had $\partial_w a = 0$, but in general this might not be possible, since a might have a pole at $z = w$, which then has to move as w does). The conclusion follows.

Exercise 3.19. Verify (3.13) directly in case $a = zwJ$.

Because of Proposition 3.30 it is sensible to ask for a family Φ_z of conformal blocks obeying

$$\partial_z(\Phi_z(\chi(z), \cdots)) = \Phi_z((\partial_z + L_{-1})\chi(z), \cdots) \tag{3.14}$$

(The point is that, if $\chi(z)$ is a section of $\mathcal{B}_K$, then the left side is zero for all z; so the right side had better also be zero, and it is, thanks to Proposition 3.30.) This can also be written in the language of $\mathcal{D}$-modules: the operator $\partial_z + L_{-1}$ makes $\mathcal{B}$ and $\mathcal{B}_K$ into $\mathcal{D}$-modules, and the sheaf of solutions Φ of (3.14) is $\mathrm{Hom}_{\mathcal{D}}(\mathcal{B}/\mathcal{B}_K, \mathcal{O})$.

Note (3.14) is equivalent to

$$\partial_z \Phi_z(\chi, \cdots) - \Phi_z(L_{-1}\chi, \cdots) = 0 \tag{3.15}$$

which might be written more briefly as $(\partial_z - L_{-1})\Phi_z = 0$. So there is a tricky minus sign here, depending whether we look at the connection on coinvariants or on conformal blocks.

In the form (3.15), our condition looks like it is describing covariantly constant sections of a connection in the space of conformal blocks. One has to be a little cautious about what this means, because the space of conformal blocks as we defined it (pointwise dual of coinvariants) does not necessarily form a bundle over C. But irrespective of this, at least we can speak meaningfully about the solutions of (3.15). In favorable situations, where $\mathcal{B}/\mathcal{B}_K$ is actually a vector bundle of finite rank, a $\mathcal{D}$-module structure really does mean a flat connection, and this is a useful intuitive picture even when it is not literally true.

Exercise 3.20. Suppose $C = \mathbb{CP}^1$. Let $v = (a+bz+cz^2)\partial_z$ be a global vector field on C. Consider a family of conformal blocks Φ_z on C with insertions at points $z_1, \dots, z_n$, obeying (3.15) in all variables. Prove that

$$\partial_v \Phi_z = (aL_{-1} + bL_0 + cL_1)\Phi_z$$

where on both sides the action is on all insertions, i.e. all z_i flow by the vector field v, and the operators $(aL_{-1} + bL_0 + cL_1)$ act on all of the inserted modules.

Example 3.32. Suppose $\mathcal{V} = V^k(\mathfrak{g})$. We take $C = \mathbb{CP}^1$, with marked points at z_1, z_2, and ∞. At z_1 and z_2 we insert parabolic Verma modules $\mathcal{R}(M_1)$ and $\mathcal{R}(M_2)$ while at ∞ we insert the irregular module $\mathcal{R}(u=0)$. The subspace $M_1 \otimes M_2 \otimes \mathbb{C}|\Omega\rangle_u \subset \mathcal{R}(M_1) \otimes \mathcal{R}(M_2) \otimes \mathcal{R}(u=0)$ represents $\mathrm{Coinv}(C)$ for every z_1, z_2. Now to write the operator $L_{-1}^{(1)}$ concretely we use the fact

$$L_{-1} = \frac{1}{k+h^\vee} \sum_{a=1}^{\dim \mathfrak{g}} (J^a_{-1} J^a_0 + J^a_{-2} J^a_1 + \cdots)$$

All positive modes annihilate M_1, so we only have to deal with $J^a_{-1} J^a_0$. Now consider the function $f = \frac{1}{z-z_1}$. We use its expansions around z_1, z_2, ∞ to see that

$$o(fJ^a) = J^a_{-1} \otimes \mathbf{1} \otimes \mathbf{1} + \sum_{n\geq 0} \frac{\mathbf{1} \otimes J^a_n \otimes \mathbf{1}}{(z_1-z_2)^{n+1}} + \sum_{n\geq 0} z_1^n (\mathbf{1} \otimes \mathbf{1} \otimes J^a_{n+1})$$

Acting on $M_1 \otimes M_2 \otimes \mathbb{C}|\Omega\rangle_u$ this reduces to

$$J^a_{-1} \otimes \mathbf{1} \otimes \mathbf{1} + \frac{\mathbf{1} \otimes J^a_0 \otimes \mathbf{1}}{z_1 - z_2}$$

Thus the image of this operator lies in $K(C)$, so modulo $K(C)$ we can replace the action of J_{-1} at z_1 with the action of J_0 at z_2. This gives

$$L^{(1)}_{-1} = -\frac{1}{k+h^\vee} \sum_{a=1}^{\dim \mathfrak{g}} \frac{J^a_0 \otimes J^a_0 \otimes \mathbf{1}}{z_1 - z_2}$$

Very similarly, if we have n marked points in the plane, the action of L_{-1} in the i-th factor is

$$L^{(i)}_{-1} = -\frac{1}{k+h^\vee} \sum_{j\neq i} \sum_{a=1}^{\dim \mathfrak{g}} \frac{(J^a_0)^{(i)} (J^a_0)^{(j)}}{z_i - z_j}$$

Altogether then, we have a PDE in the bundle $\mathrm{Coinv}(C) \simeq \bigotimes_i M_i$ over $\mathrm{Conf}_n(\mathbb{C})$, given by the operators

$$\partial_{z_i} - \frac{1}{k+h^\vee} \sum_{j\neq i} \frac{T_{ij}}{z_i - z_j} \tag{3.16}$$

where $T_{ij} = \sum_{a=1}^{\dim \mathfrak{g}} (X^a)^{(i)} (X^a)^{(j)}$.

The monodromy of this PDE gives an action of the pure braid group $P_n = \pi_1(\mathrm{Conf}_n(\mathbb{C}))$ on $\bigotimes_i V_i$. What is this action concretely? It is easiest to discuss in the case $n = 2$; there it is generated by a single loop. The monodromy of the equation

$$\partial_z - \frac{1}{k+h^\vee}\frac{T}{z}$$

when z goes around 0 in the positive direction is just given by the automorphism

$$\exp\left(\frac{2\pi\mathrm{i}}{k+h^\vee}T\right) \in \mathrm{End}(V_1 \otimes V_2)$$

In case $n = 3$ it is already more interesting. The local monodromy around $z_i = z_j$ will be conjugate to the automorphism $\exp\left(\frac{2\pi\mathrm{i}}{k+h^\vee}T_{ij}\right)$ as long as we are in the "non-resonant" situation, where no two eigenvalues of $\frac{T_{ij}}{k+h^\vee}$ differ by an integer. But there are also the change-of-basis maps between the eigenbases at different loci $z_i = z_j$, which are hard to write explicitly except for $\mathfrak{g} = \mathfrak{sl}_2$.

There is another source of braid group representations here, namely the theory of the quantum universal enveloping algebra $U_q(\mathfrak{g})$, which provides for every V, W an *R-matrix* $R_{VW} \in \mathrm{End}(V \otimes W)$ obeying on $V_1 \otimes V_2 \otimes V_3$ the relation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}\,,$$

and compatible with morphisms and tensor products (in an appropriate sense, so-called *hexagon relation*). For example, if $\mathfrak{g} = \mathfrak{sl}_2$ and V, W are both the fundamental representation, then R_{VW} is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 1-q^2 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Given this R-matrix and representations $V_1, \ldots, V_n$ of $\mathfrak{g}$, one can construct a representation of the pure braid group P_n on $\bigotimes_i V_i$. Then we have:

Theorem 3.33 (Drinfeld-Kohno). For generic k, the representation of P_n on $\otimes_{i=1}^n V_i$ coming from monodromy of coinvariants for $V^k(\mathfrak{g})$ is equivalent to the one coming from $U_q(\mathfrak{g})$, with the relation $q = \exp(\frac{\pi\mathrm{i}}{k+h^\vee})$.

Note it really says *equivalent* and not *equal*. You couldn't really ask for more here, since the representation given by monodromy of coinvariants depends on a choice of basepoint and generating loops. (But we will say something better in a bit.)

Here "generic k" is supposed to mean there are no resonances. (Apparently there are different versions of the statement, e.g. Drinfeld's version is really an $\hbar$-adic statement with $\hbar = 1/(k+h^\vee)$).

For a quick check, looking up the R-matrices one sees that the operator $R_{12} \circ R_{12}$ on $V_1 \otimes V_2$ acts as $q^{2T_{12}}$, matching with the monodromy of coinvariants which we discussed above. This proves the Drinfeld-Kohno theorem in case $n = 2$.

Exercise 3.21. Prove directly that the equations (3.16) define a *flat* connection, i.e. they are a compatible system of PDEs.

Exercise 3.22. In the above we derived the equation (3.16) from the connection on coinvariants when we inserted the module $\mathcal{R}(u = 0)$ at $z = \infty$. What equation do we get if we instead insert $\mathcal{R}(u)$ for some generic u?

Exercise 3.23. Consider the case of $\mathfrak{g} = \mathfrak{sl}_2$, with $n = 2$, and $z_1 = 0$, $z_2 = 1$. Let V_1 and V_2 both be the fundamental representation, V. Then we have two operators $b_\pm$ on $\mathrm{Coinv}(C) \simeq V \otimes V$ corresponding to exchanging z_1 and z_2: either we move z_1 in the upper half-plane and z_2 in the lower, or vice versa. Prove that these operators obey the "Hecke relation" [check]

$$b_+ - b_- = (q - q^{-1})\mathbf{1}\,.$$

What is the analogous relation if we take $\mathfrak{g} = \mathfrak{sl}_N$?

3.9. **Duals.** Given a $\mathcal{V}$-module $\mathcal{R}$, we can define its *dual* in a tautological way: we imagine putting $\mathcal{R}$ at 0 on the sphere, and then the dual consists of all objects which can be placed at ∞.

Definition 3.34 (Dual module). Say $\mathcal{V}$ is a conformal vertex algebra, and let $\mathcal{R}$ be an ordinary $\mathcal{V}$-module. Then let $\mathcal{R}^* = \bigoplus \mathcal{R}_n^*$ be the graded dual space,[10] and define $Y^{\mathcal{R}^*}(a,z) \in \mathrm{End}\,\mathcal{R}^*[[z]]$ by

$$(Y^{\mathcal{R}^*}(a,z)\alpha)\cdot\chi = \alpha\cdot(Y^{\mathcal{R}}(\mathrm{e}^{zL_1}(-z^{-2})^{L_0}a, z^{-1})\chi)\,. \tag{3.17}$$

To motivate this definition recall that around a point $z \in \mathbb{C}^\times$ there are two natural pointed coordinate systems: $t(w) = w - z$ and $t'(w) = 1/w - 1/z$. The transformation $g(z) = \mathrm{e}^{zL_1}(-z^{-2})^{L_0}$ appearing in (3.17) is the action of the change of coordinates from t to t'. Then (3.17) is

$$(Y^{\mathcal{R}^*}(a,z)\alpha)\cdot\chi = \alpha\cdot(Y^{\mathcal{R}}(\rho(g(z))a, z^{-1})\chi)\,. \tag{3.18}$$

[prove that it is really a vertex algebra module]

In the case of $\mathcal{V} = V^k(\mathfrak{g})$ it becomes more familiar:

Example 3.35. If $\mathcal{V} = V^k(\mathfrak{g})$ then (3.17) implies

$$(Y^{\mathcal{R}^*}(J^X,z)\alpha)\cdot\chi = \alpha\cdot(Y^{\mathcal{R}}(-z^{-2}J^X, z^{-1})\chi)$$

so

$$J_n^X\alpha\cdot\chi + \alpha\cdot J_{-n}^X\chi = 0 \tag{3.19}$$

and conversely checking (3.19) is sufficient to verify (3.17).

Exercise 3.24. Consider the involution $\lambda \mapsto \lambda^* = -w_0(\lambda)$, where w_0 is the longest element in the Weyl group $W(\mathfrak{g})$. (For $\mathfrak{g} = \mathfrak{sl}_2$ we just have $\lambda^* = \lambda$.) Let L_λ be the finite-dimensional $\mathfrak{g}$-module with highest weight λ. Then L_{λ^*} is the dual of L_λ. Prove similarly that the dual of $\mathcal{R}(L_\lambda)$ is $\mathcal{R}(L_{\lambda^*})$, provided that k is generic (so that the invariant bilinear form on $\mathcal{R}(L_\lambda)$ is nondegenerate).

Exercise 3.25. Prove that the dual pairing $\Phi : \mathcal{R}\otimes\mathcal{R}^* \to \mathbb{C}$ is a conformal block on $\mathbb{CP}^1$ with $\mathcal{R}$ inserted at 0 and $\mathcal{R}^*$ inserted at ∞.

Exercise 3.26. Prove more generally that on $\mathbb{CP}^1$ with $\mathcal{R}_1$ and $\mathcal{R}_2$ inserted the space of conformal blocks is $\mathrm{Hom}(\mathcal{R}_1, \mathcal{R}_2^*)$ (where Hom means the space of morphisms of $\mathcal{V}$-modules).

Exercise 3.27. Prove that $(\mathcal{R}^*)^* = \mathcal{R}$ in a canonical way.

Exercise 3.28. Prove that $\mathcal{R} \simeq \mathcal{R}^*$ if and only if there exists a nondegenerate bilinear pairing $\Phi : \mathcal{R}\otimes\mathcal{R} \to \mathbb{C}$ which is "invariant" in the sense that

$$\Phi(Y(a,z)\chi,\chi') = \Phi(\chi, Y(\mathrm{e}^{zL_1}(-z^{-2})^{L_0}a, z^{-1})\chi')\,.$$

[10] Some authors would call this $\mathcal{R}'$ and reserve $\mathcal{R}^*$ for the vector space dual without regard for grading.

Exercise 3.29. Prove that the space of invariant bilinear pairings in $\mathcal{V}$ is $(\mathcal{V}_0/L_1\mathcal{V}_1)^*$.

Since $\mathcal{V}_0 = \mathbb{C}|\Omega\rangle$ this means that there is a unique invariant bilinear pairing up to scale as long as L_1 annihilates $\mathcal{V}_1$. Then we have $\mathcal{R} \simeq \mathcal{R}^*$ if and only if this unique pairing is nondegenerate.

Exercise 3.30. Prove that $\mathrm{Hom}(\mathcal{R}_1, \mathcal{R}_2) = \mathrm{Hom}(\mathcal{R}_2^*, \mathcal{R}_1^*)$.

3.10. **Varying the curve.** So far we discussed how to vary conformal blocks when we move marked points, i.e. we got a connection on the bundle of conformal blocks over C. But the Virasoro action on $\mathcal{V}$ actually allows us to do more: we can vary the curve C itself, thus obtaining a connection over the moduli space of curves (or more precisely a slightly twisted connection as we discuss below.)

Deformations of C are realized by the cohomology $H^1(C, T)$. The details of how we describe this connection depends on exactly how we realize this cohomology.

(1) Suppose given a Cech cocycle v, i.e. cover C by patches U_α and give vector fields $v_{\alpha\beta}$ on overlaps $U_{\alpha\beta}$, obeying $v_{\alpha\beta} + v_{\beta\gamma} + v_{\gamma\alpha} = 0$. Now define contours $c_{\alpha\beta}$ as follows. Choose an ordering $<$ arbitrarily on the index set. Shrink each U_α slightly to V_α with smooth boundary, such that they still cover C, and the boundaries meet transversely. Then define disjoint regions

$$R_\alpha = V_\alpha \setminus \bigcup_{\gamma<\alpha} V_\gamma.$$

For each $\alpha < \beta$, let $c_{\alpha\beta}$ be the common boundary

$$c_{\alpha\beta} = \partial R_\alpha \cap \partial R_\beta,$$

with the boundary orientation induced by R_α. Then the variation of chiral correlation

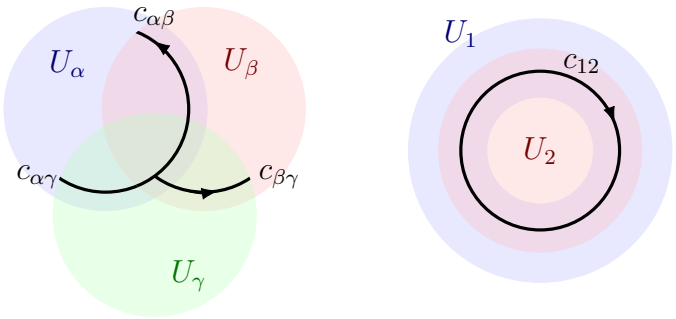


FIGURE 1. Cover by patches U_α and shrunken sets. After choosing an ordering on the cover, here $\alpha < \beta < \gamma$, the sets $R_\alpha = V_\alpha \setminus \bigcup_{\gamma<\alpha} V_\gamma$ give a partition of the surface, and the contours $c_{\alpha\beta}$ are the interfaces between R_α and R_β, oriented as boundary components of R_α.

functions induced by v is

$$\partial_v \langle \cdots \rangle = \frac{1}{2\pi \mathrm{i}} \sum_{\alpha<\beta} \int_{z \in c_{\alpha\beta}} v_{\alpha\beta}(z) \langle T(z) \cdots \rangle \tag{3.20}$$

A simple but useful example arises when we cover C by two open sets $U_{1,2}$, intersecting in an annulus. Then the deformation is induced by a single vector field v_{12} on the annulus,

and (3.20) becomes an integral over a loop c_{12} around the annulus:

$$\partial_v \langle \cdots \rangle = \oint_{z \in c} v_{12}(z) \langle T(z) \cdots \rangle$$

(2) Suppose given an infinitesimal Beltrami differential $\mu \in \Omega^{0,1}(T)$, corresponding to an infinitesimal variation of the complex structure on C.[11] Then

$$\partial_\mu \langle \cdots \rangle = \frac{1}{2\pi \mathrm{i}} \int_{z \in C} \mu(z) \langle T(z) \cdots \rangle \tag{3.21}$$

The formulas (3.21) and (3.20) are related essentially by integration by parts. Indeed, given the cocycle $v_{\alpha\beta}$ a corresponding Beltrami differential can be given as follows. Choose a smooth partition of unity (ρ_γ) subordinate to the open cover (U_γ), and define smooth local vector fields

$$\xi_\alpha = \sum_\gamma \rho_\gamma v_{\alpha\gamma}$$

on each patch U_α. Since $(v_{\alpha\beta})$ is a Cech cocycle, these obey

$$\xi_\alpha - \xi_\beta = v_{\alpha\beta}$$

on the overlaps $U_{\alpha\beta}$. Therefore the local $(0,1)$-forms

$$\mu|_{U_\alpha} = \bar\partial \xi_\alpha = \sum_\gamma (\bar\partial \rho_\gamma) v_{\alpha\gamma}$$

glue to a global Beltrami differential $\mu \in \Omega^{0,1}(T)$. Now, to compare the variation formulas, write $F(z) = \langle T(z) \cdots \rangle$. Since the support of the deformation avoids the insertions, $F(z)$ is holomorphic on a neighborhood of each R_α. Hence on R_α we have

$$\mu(z) F(z) = \bar\partial(\xi_\alpha(z) F(z)).$$

Applying Stokes' theorem on each R_α gives

$$\int_C \mu(z) F(z) = \sum_\alpha \int_{R_\alpha} \bar\partial(\xi_\alpha F) = \sum_\alpha \int_{\partial R_\alpha} \xi_\alpha(z) F(z).$$

The internal boundary between R_α and R_β with $\alpha < \beta$ is the contour $c_{\alpha\beta}$. It contributes once with the orientation induced from R_α and once with the opposite orientation induced from R_β; the two terms combine to give

$$\int_{z \in c_{\alpha\beta}} (\xi_\alpha(z) - \xi_\beta(z)) F(z) = \int_{z \in c_{\alpha\beta}} v_{\alpha\beta}(z) F(z).$$

Summing over all interfaces and using that C has no outer boundary, we get

$$\frac{1}{2\pi \mathrm{i}} \int_C \mu(z) \langle T(z) \cdots \rangle = \frac{1}{2\pi \mathrm{i}} \sum_{\alpha < \beta} \int_{z \in c_{\alpha\beta}} v_{\alpha\beta}(z) \langle T(z) \cdots \rangle,$$

recovering (3.20).

Note that both formulas are well defined only as long as the support of $\mu(z)$ or $v_{\alpha\beta}(z)$ avoids the points where the insertions $\cdots$ are placed. This is a good feature, not a bug; it allows us to make canonical sense of what we mean by varying the curve while "holding the insertions fixed." Every deformation class can be represented by $\mu(z)$ or $v_{\alpha\beta}(z)$ avoiding any particular set of points.

[11] One fast way to say it: deform $\bar\partial \to \bar\partial - \epsilon\mu\partial$; since $\bar\partial\mu = 0$ this still gives a complex structure to first order in ϵ.

Using either approach, there is something to prove: we need to check that this variation procedure actually maps conformal blocks to conformal blocks, i.e. that it preserves the condition of vanishing on coinvariants. We formulate the statement in the Cech language.

Proposition 3.36. Let $\mathcal{M}$ be a moduli space parameterizing deformations of C, and let s be a section of the bundle $\mathcal{B}_K$ over $\mathcal{M}$. Let $v \in T_C\mathcal{M} = H^1(C, T)$, let $(v_{\alpha\beta})$ be a Cech representative for v, and let $T_v \in \mathrm{End}(\bigotimes_i \mathcal{R}_i)$ be the operator induced by

$$\frac{1}{2\pi \mathrm{i}} \sum_{\alpha<\beta} \int_{z \in c_{\alpha\beta}} v_{\alpha\beta}(z) T(z)\,.$$

(Note this operator is defined only modulo $K(C)$.) Then $(\partial_v + T_v)s \in K(C)$.

Here are some ideas of the proof. To make the notation easier, it is convenient to consider a special kind of representative:[12] we take just two open sets, U_1 and U_2, where U_1 is a small coordinate disc around a marked point p, and U_2 is $C \setminus \{p\}$, and we take v_{12} to be a meromorphic vector field on the punctured disc U_{12}. Then we have a canonical representative for T_v, obtained by mode expansion at the marked point p.

Now let a be a section of $\mathcal{V}_C$. We want to extend a to nearby deformations of C, but we are not too ambitious: we try to extend only to first order along the deformation v. Then the first-order variation $\partial_v a$ is still given by a section of $\mathcal{V}_C$. (It may have a higher-order pole at p than a has; think of the example of $\partial_w(1/(z-w)) = 1/(z-w)^2$, where at $w = 0$ we have $a = 1/z$ but $\partial_w a = -1/z^2$. The general condition is that $\partial_v a - \mathcal{L}_{v_{12}} a$ should be regular around p.) Then we compute directly. Write in the local coordinate around p

$$a(z) = \sum_{m\in\mathbb{Z}} a_m z^m \,\mathrm{d}z, \qquad \mathcal{L}_{v_{12}} a(z) = \sum_{m\in\mathbb{Z}} \ell_m z^m \,\mathrm{d}z,$$

so by definition

$$o(a) = \sum_m (a_m)_{(m)}, \qquad o(\mathcal{L}_{v_{12}} a) = \sum_m (\ell_m)_{(m)}.$$

For T_v (the mode expansion of $\frac{1}{2\pi\mathrm{i}} \oint v_{12}(z)T(z)$ at p), the same contour manipulation used above for (3.13), now with the weight $v_{12}(z)$ in the contour integral, gives

$$[T_v, o(a)] = -\,o(\mathcal{L}_{v_{12}} a)\,.$$

Also, differentiating $o(a) = \sum_m (a_m)_{(m)}$ along the deformation[13],

$$[\partial_v, o(a)] = \partial_v(o(a)) = o(\partial_v a) + o(\mathcal{L}_{v_{12}} a)\,.$$

(Here the extra $o(\mathcal{L}_{v_{12}} a)$ term is exactly the same "moving-pole" term as in (3.13); only L_{-1} is replaced by the full generator T_v.) Adding the two identities gives

$$[\partial_v + T_v, o(a)] = o(\partial_v a)\,.$$

We actually proved a special case of this earlier, in (3.13). That is the case where v_{12} is the constant vector field ∂_z.

[12] Although I said that this is a special representative, actually every deformation can be realized in this way: this fact is called *Virasoro uniformization*.

[13] Write the first-order deformation as a map of local coordinates $z \mapsto z + \epsilon\, v_{12}(z)$. Then $a(z + \epsilon v_{12}(z)) = a(z) + \epsilon\, \mathcal{L}_{v_{12}} a(z) + O(\epsilon^2)$, so the coefficient of $z^m \mathrm{d}z$ in the deformed expansion acquires two contributions: one from the intrinsic variation $\partial_v a$, and one from re-expanding in the fixed coordinate z (the same moving-pole term as in (3.13)). Applying $o(\cdot)$ to these first-order coefficients gives exactly $\partial_v(o(a)) = o(\partial_v a) + o(\mathcal{L}_{v_{12}} a)$.

[...]

3.11. **Virasoro twisting.** One caution: the formula (3.21) depends on the local coordinate we use. If $T(z)$ transformed as a quadratic differential, then $\mu(z)T(z)$ would be a $(1,1)$-form, perfect for integration. But the truth is that $T(z)$ is not quite a quadratic differential, rather it has the transformation law (3.12). Using that formula we see that changing the local coordinate $z \to g(z)$ changes the integrand in (3.21) by

$$\frac{1}{2\pi \mathrm{i}} \frac{c}{12} \mu(z) \{g(z), z\} \langle \cdots \rangle$$

There are a few ways of thinking about this problem. One, the most concrete, is to fix *complex projective structures* on our surfaces:

Definition 3.37 (Complex projective structure). A *complex projective structure* S on a Riemann surface C is an atlas of local coordinate systems, for which all transition maps are Möbius transformations.

Every surface admits a complex projective structure, e.g. from Fuchsian uniformization; indeed Fuchsian uniformization gives a complex projective structure where the transition maps are in $\mathrm{PSL}(2,\mathbb{R})$, not only $\mathrm{PSL}(2,\mathbb{C})$. But for our purposes we don't want to use that one: we want to use one which varies holomorphically as we vary C (i.e. there should exist coordinate systems in the atlas which vary holomorphically as functions of the moduli of C).

Given two complex projective structures we can consider their *difference*:

Definition 3.38 (Difference of complex projective structures). Let S_1 and S_2 be two complex projective structures on C. There is a globally defined holomorphic quadratic differential $S_2 - S_1$ on C, given locally by $S_2 - S_1 = \{w, z\}\,\mathrm{d}z^2$, where z is a local coordinate in S_1 and w is a coordinate in S_2.

Proposition 3.39. For any smooth compact Riemann surface C, the complex projective structures on C form a torsor over the space $H^0(C, K^2)$ of holomorphic quadratic differentials.

Thus complex projective structures make up a holomorphic affine bundle over the moduli space $\mathcal{M}$ of curves. Once we locally fix a holomorphic section S of the bundle of complex projective structures, (3.21) defines an honest connection locally over the moduli of curves. The difference $S_1 - S_2$ is a section of the bundle of holomorphic quadratic differentials over $\mathcal{M}$. This bundle is also $T^*\mathcal{M}$: indeed we have a pairing between Beltrami differentials μ and holomorphic quadratic differentials ϕ_2 given by $\int_C \mu\phi_2$. Thus, we can think of $S_1 - S_2$ as a 1-form λ on $\mathcal{M}$.

Altogether, then, for each choice of S we get a covariant derivative operator ∇^S in the sheaf of conformal blocks over $\mathcal{M}$, with the rule that

$$\nabla^{S+\lambda} = \nabla^S + \frac{c}{24\pi \mathrm{i}} \lambda\,.$$

[calculate the curvature]

In particular, recall that $\pi_1(\mathcal{M})$ is the mapping class group of C. Also, suppose we are in the nicest possible situation, where the spaces of conformal blocks are finite-dimensional, so that we can find covariantly constant sections of ∇^S locally. Then, were it not for the twisting, parallel transport of covariantly constant sections would produce a representation of $\pi_1(\mathcal{M})$. Because of the twisting, we get something slightly different: we get a *projective* representation, i.e. a representation of a *central extension* of $\pi_1(\mathcal{M})$.

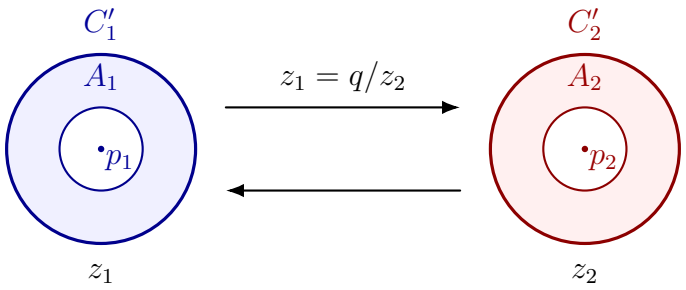


FIGURE 2. Plumbing: remove small discs around p_1, p_2 to get C_1', C_2' with annuli A_1, A_2, then identify by $z_1 \sim q/z_2$ to form C_q.

3.12. **Plumbing conformal blocks.**

Definition 3.40 (Plumbing Riemann surfaces). Suppose C_1 and C_2 are two Riemann surfaces with marked points p_1, p_2. Fix holomorphic coordinates z_1, z_2 around these points. Also fix parameters $r_1, r_2 > 0$ such that the z_i-coordinate patch contains $\{|z_i| < r_i\}$, and fix $q \in \mathbb{C}^\times$ with $0 < |q| < r_1 r_2$. Let

$$C_1' = C_1 \setminus \{|z_1| \le |q|/r_2\}, \qquad C_2' = C_2 \setminus \{|z_2| \le |q|/r_1\}\,.$$

Then we have annuli

$$A_1 = \{|q|/r_2 < |z_1| < r_1\} \subset C_1', \qquad A_2 = \{|q|/r_1 < |z_2| < r_2\} \subset C_2'\,.$$

Finally let $C_q = (C_1' \sqcup C_2')/\sim$, where $\sim$ is the relation

$$z_1 \sim q/z_2, \qquad z_1 \in A_1, \quad z_2 \in A_2\,.$$

Exercise 3.31. Prove that the plumbed surface C_q does not depend on the choice of the r_i, and that it depends on the local coordinates only through their 1-jets at the marked points; rescaling the local coordinate on either side rescales q. Said more abstractly, we use a bilinear pairing b between the tangent spaces at the marked points; then identifying each side locally with its tangent space, the plumbed surface is $b(z_1, z_2) = q$.

One can also think of plumbing as connecting the two sides $C_i'' = C_i \setminus \{|z_i| \le r_i\}$ by adding in a cylinder, whose modulus is $|q|/r_1 r_2$. Conformally, this cylinder is equivalent to a tube with circumference 2π and length $\log(r_1 r_2/|q|)$. In the limit $q \to 0$ this cylinder becomes infinitely long, and the plumbed surface develops a node.

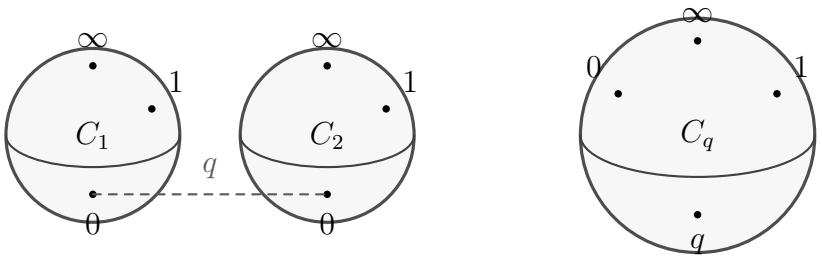


FIGURE 3. After plumbing two thrice-marked spheres at 0 and 0, the resulting sphere has four marked points with cross-ratio q.

Exercise 3.32. Suppose that C_1 and C_2 are spheres, with the standard coordinates z_1 and z_2, each with marked points at $0, 1, \infty$. Fix some q with $|q| < 1$. Plumbing together the marked points at $z_1 = 0$ and $z_2 = 0$, the plumbed surface C_q is again a sphere, with four marked points. Prove that the invariant cross-ratio of these four points is q.

Now we claim it's possible to make conformal blocks on the plumbed surface from conformal blocks on the original two, as follows.

Suppose given conformal blocks Φ_1 on C_1 and Φ_2 on C_2, with $\mathcal{R}$, $\mathcal{R}^*$ inserted at marked points p_1, p_2, and possibly other insertions χ_1 on C_1, χ_2 on C_2. Then let h be the minimum conformal weight in $\mathcal{R}$, and define

$$\Phi(\chi_1, \chi_2) = \sum_{n \geq h} q^n \sum_i \Phi_1(\chi_1, \xi_{i,n}) \Phi_2(\xi^*_{i,n}, \chi_2) \tag{3.22}$$

where $(\xi_{i,n})$ and $(\xi^*_{i,n})$ are dual bases for $\mathcal{R}_n$ and $\mathcal{R}^*_n$, with respect to the invariant pairing.[14] Note that in the limit $q \to 0$ the leading term comes from the primaries of weight h in $\mathcal{R}$ contribute, and if we rescale by an overall factor q^{-h} then these are the only ones that survive the limit; for finite q the whole tower of descendants is involved.

The formula (3.22) makes sense for $q \in \mathbb{R}_+$. If the conformal weights $n \notin \mathbb{Z}$, then it is multivalued as a function of $q \in \mathbb{C}$. To make it well defined we can replace $q \to \exp(2\pi i \tau)$; then we get a single-valued function of τ.

Here is the idea of the definition (3.22). Suppose we consider a completely arbitrary linear functional of χ_1 and χ_2. We could try to build chiral correlation functions out of such a functional, and they will make sense at least on formal discs around the insertions. If our functional is of the form $\Phi_1(\chi_1, \xi)\Phi_2(\xi', \chi_2)$ then the chiral correlation functions will make sense (obeying all the usual properties) on $C_1 \cup C_2$ with the punctures p_1, p_2 deleted. More generally the same is true if we take any $\sum_j \Phi_1(\chi_1, \xi_j)\Phi_2(\xi'_j, \chi_2)$. Now, what we want to do is to engineer the object $\mathcal{I} = \sum_j \xi_j \otimes \xi'_j$ in such a way that the chiral correlation functions we obtain on the annuli A_1 and A_2 are the same, and thus can be glued to give chiral correlation functions on C. This will happen if we choose

$$\mathcal{I} = \sum_{n \geq h} q^n \sum_i \xi_{i,n} \otimes \xi^*_{i,n} = \sum_{n \geq h} \sum_i (q^{L_0} \xi_{i,n}) \otimes \xi^*_{i,n} \,.$$

Indeed, suppose we insert $a(z_1)$ on C_1. Expanding near $z_1 = 0$ we see that inside the correlation function this replaces $\mathcal{I}$ by

$$\sum_{n,i} (Y^{\mathcal{R}}(a, z_1) q^{L_0} \xi_{i,n}) \otimes \xi^*_{i,n} \,. \tag{3.23}$$

On the other hand, if we insert $a(z_2)$ near $z_2 = 0$ on C_2 we get

$$\sum_{n,i} q^{L_0} \xi_{i,n} \otimes (Y^{\mathcal{R}^*}(a, z_2) \xi^*_{i,n}) \,.$$

Using (3.18), this is

$$\sum_{n,i} q^{L_0} (Y^{\mathcal{R}}(\rho(g(z_2))a, z_2^{-1}) \xi_{i,n}) \otimes \xi^*_{i,n}$$

[14] Or, without choosing a basis, one could say we insert the operator q^{L_0} into the $\mathcal{R} \otimes \mathcal{R}^*$ slots of $\Phi_1 \otimes \Phi_2$.

with $g(z_2)$ the pointed version of the transformation $z \to 1/z$. Commuting the q^{L_0} through we get

$$\sum_{n,i} (Y^{\mathcal{R}}(\rho(g(z_2,q))a, qz_2^{-1}) q^{L_0} \xi_{i,n}) \otimes \xi_{i,n}^* \tag{3.24}$$

where $g(z_2, q)$ is the pointed version of $z \to q/z$. Finally, using the coordinate covariance of $Y^{\mathcal{R}}$ and $z_1 = q/z_2$ we see that (3.23) and (3.24) match, as desired.

It is not at all obvious that the sum (3.22) converges, but at least *formally* it gives a conformal block:

Exercise 3.33. Assuming that the sum in (3.22) converges, prove that it gives a conformal block on the plumbed surface C.

More generally, instead of having two surfaces we can just suppose C_0 is a surface (either disconnected or connected), with two punctures carrying $\mathcal{R}$ and $\mathcal{R}^*$, and a conformal block; then we can plumb those two punctures together and get a conformal block on the plumbed surface C_q.

Exercise 3.34. Suppose $C_0 = \mathbb{CP}^1$ with two punctures at $z = 0$ and $z = \infty$ carrying $\mathcal{R}$ and $\mathcal{R}^*$ respectively. Then there is a canonical block on C_0 given by the dual pairing. Show that by plumbing we get a block on the torus $C_q = \mathbb{C}^\times/(z \sim qz)$, and in this block

$$\langle 1 \rangle = \mathrm{Tr}_{\mathcal{R}}\, q^{L_0}\,.$$

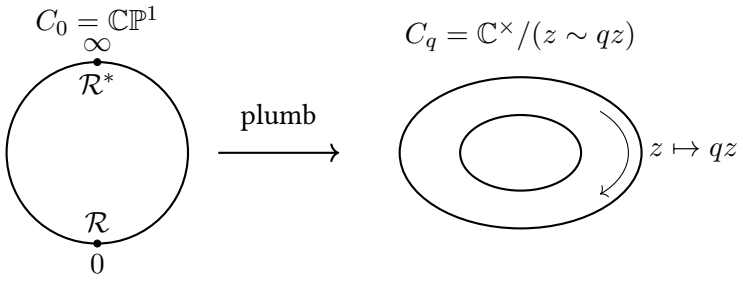


FIGURE 4. Plumbing the punctures on $\mathbb{CP}^1$ produces the torus C_q.

Exercise 3.35. Suppose C_1 is a surface with a puncture at p_1 carrying $\mathcal{R}$, and a conformal block. Let C_2 be a copy of $\mathbb{CP}^1$ carrying $\mathcal{R}^*$ at $p_2 = 0$ and $\mathcal{R}$ at ∞. Now plumb C_1 and C_2, gluing p_1 to p_2. What is the resulting conformal block? [I bet it's the original block on C_1 again but we should check!]

Finally let us try to formulate this construction in a more intrinsic way. Recall that in the coordinate-invariant framework, to define conformal blocks with general ordinary modules $\mathcal{R}$, we need to fix a tangent vector at each marked point. Then, suppose given a surface C_0 with two marked points and two tangent vectors. These two vectors can be used to define conformal blocks with $\mathcal{R}$ and $\mathcal{R}^*$ inserted; simultaneously they determine a bilinear pairing, which we use for plumbing. [...]

We can (and should) also ask whether the gluing operation is compatible with the connection coming from variation of the curve. The answer is yes, *if* we choose a complex projective structure which includes the glued coordinate z_1 or z_2. Here is the concrete check in the special case of gluing two genus-0 components.

Proposition 3.41 (Compatibility of plumbing with the Virasoro connection, genus 0 case). In the genus-0 plumbing setup above, assume the blocks Φ_1 and Φ_2 are flat for the Virasoro connection on their respective components (in the sense of (3.15) and Exercise 3.20). Let the insertions on C_1 away from the node be at $x_1, \dots, x_r$. Then the plumbed block Φ_q from (3.22) is flat for the induced q-direction Virasoro connection. Equivalently,

$$\left(q\partial_q - \sum_{a=1}^{r} x_a \partial_{x_a} \right) \Phi_q = 0, \tag{3.25}$$

and, using (3.15) in each x_a-direction, this is equivalently

$$q\partial_q \Phi_q = \sum_{a=1}^{r} x_a \, \Phi_q(\chi_{1,1}, \dots, L_{-1}\chi_{1,a}, \dots, \chi_{1,r}; \chi_2). \tag{3.26}$$

So the q-direction connection is expressed purely via L_{-1}.

[check this prop and proof carefully]

Proof. Write (3.22) as

$$\Phi_q(\chi_1, \chi_2) = \sum_{n \geq h} \sum_{i} q^n \Phi_1(\chi_1, \xi_{i,n}) \Phi_2(\xi^*_{i,n}, \chi_2).$$

Differentiating term-by-term gives

$$q\partial_q \Phi_q = \sum_{n,i} n q^n \Phi_1(\chi_1, \xi_{i,n}) \Phi_2(\xi^*_{i,n}, \chi_2).$$

Now use Exercise 3.20 on the component block Φ_1, before gluing: Φ_1 is a conformal block on the sphere C_1 with punctures $x_1, \dots, x_r$ and the node puncture at $z_1 = 0$, so the exercise applies directly to that sphere. We apply it with $v = z_1\partial_{z_1}$. The node insertion on C_1 is at coordinate $u = 0$, and the geometric part of the derivative at a puncture is $v(u)\partial_u$; here $v(0) = 0$, so the node contributes no position-derivative term. For a homogeneous node state $\xi_{i,n}$ of degree n, the action of this infinitesimal dilation at the node is multiplication by n. Hence, with $\xi = \xi_{i,n}$,

$$\sum_{a=1}^{r} x_a \partial_{x_a} \Phi_1(\chi_1, \xi_{i,n}) = \sum_{a=1}^{r} \Phi_1(\chi_{1,1}, \dots, z\partial_z \chi_{1,a}, \dots, \chi_{1,r}, \xi_{i,n}) + n\, \Phi_1(\chi_1, \xi_{i,n}).$$

Now multiply by $q^n \Phi_2(\xi^*_{i,n}, \chi_2)$ and sum over (n, i). Comparing with the expression for $q\partial_q \Phi_q$ above yields

$$q\partial_q \Phi_q = \sum_{a=1}^{r} x_a \partial_{x_a} \Phi_q,$$

which is (3.25). Finally, use (3.15) in each x_a direction to rewrite ∂_{x_a} by insertion of L_{-1}, obtaining (3.26). So Φ_q is flat in the plumbing direction. □

The same argument also gives flatness in the other moduli directions. For instance, for a puncture position x_a on C_1,

$$\partial_{x_a} \Phi_q = \sum_{n,i} q^n \left(\partial_{x_a} \Phi_1(\chi_1, \xi_{i,n}) \right) \Phi_2(\xi^*_{i,n}, \chi_2)$$

and because Φ_1 is flat we can replace ∂_{x_a} by the corresponding $L_{-1}^{(a)}$ action using (3.15). Thus Φ_q is flat in each x_a direction. Likewise, for puncture positions on C_2, flatness of Φ_2 implies flatness of Φ_q term-by-term.

Another popular choice would be to write $z = \exp(2\pi i t)$ and use the coordinate t. In that case we need to include an extra $q^{-c/24}$ in the plumbing formula to get it to be compatible. (The difference comes from the fact that the Schwarzian derivative of the change of coordinates from t to z is $\{t, z\} = \frac{1}{2}$, so the Virasoro field in the t-coordinate has an extra $c/24$.)

3.13. **Unitarity.** So far all the structures we have discussed were complex-linear. But in addition, we may have another symmetry, which introduces some useful *positivity* in the story. In this section we let $\overline{V}$ denote the complex conjugate vector space to V.

Suppose $\mathcal{V}$ is a conformal vertex algebra, and equip $\mathcal{V}$ with a map $\sigma : \mathcal{V} \to \overline{\mathcal{V}}$, such that $\sigma^2 = \mathbf{1}$ (sometimes called PCT symmetry). For instance, if $\mathcal{V} = V^k(\mathfrak{g})$ one way to get such a σ would be to choose a real structure on $\mathfrak{g}$.

Then we can define:

Definition 3.42 (Conjugate $\mathcal{V}$-module). Suppose $\mathcal{R}$ is an ordinary $\mathcal{V}$-module. Then we make $\overline{\mathcal{R}}$ into a $\mathcal{V}$-module by the action

$$Y^{\overline{\mathcal{R}}}(a, z)\overline{\chi} = \overline{Y^{\mathcal{R}}(\sigma(a), \overline{z})\chi}\,.$$

Definition 3.43 (Hermitian pairing). A *Hermitian pairing* on $\mathcal{R}$ is an isomorphism $\overline{\mathcal{R}} \simeq \mathcal{R}^*$, where $\mathcal{R}^*$ is the dual module defined above. Such a pairing induces a Hermitian pairing $(\cdot, \cdot)$ on the vector space $\mathcal{R}$.

Definition 3.44 (Unitarity). We call a $\mathcal{V}$-module $\mathcal{R}$ *unitary* if it is equipped with a positive definite Hermitian pairing.

The Heisenberg vertex algebra can be made unitary, but as far as I understand this is not true for more general $V^k(\mathfrak{g})$ or for Virasoro. But we will discuss some more interesting unitary vertex algebras below.

4. The rational case

4.1. **Category of ordinary modules.** So far we have been working with the vertex algebra $\mathcal{V}^k(\mathfrak{g})$ at generic k. Now let us instead take $k \in \mathbb{N}$ and consider the simple quotient $\mathcal{V} = L_k(\mathfrak{g})$. In this case some things become simpler. In particular, the category of modules becomes simpler.

Definition 4.1. An *ordinary $\mathcal{V}$-module* is a $\mathcal{V}$-module for which L_0 acts diagonalizably, with eigenvalues bounded below, and finite-dimensional eigenspaces.

Let $\mathcal{C}$ be the category of ordinary modules for $L_k(\mathfrak{g})$.

4.2. **The premodular tensor category structure.** We will explain in this section that $\mathcal{C}$ has a bunch of nice structures, which together make it a *premodular tensor category*.

4.2.1. *Semisimplicity.*

Proposition 4.2 (Nice properties of ordinary modules for $L_k(\mathfrak{g})$). For $k \in \mathbb{N}$, every ordinary module for $L_k(\mathfrak{g})$ is a finite direct sum of simple ones. ("Simple" means they have no proper submodules.) The simple ones are irreducible quotients of parabolic Verma modules, with highest

weights λ subject to the condition $\lambda \cdot \theta^\vee \leq k$, where $\theta^\vee \in \mathfrak{h}$ is the highest coroot of $\mathfrak{g}$. Call these simple modules $\mathcal{R}(\lambda)$.

In particular: $\mathcal{C}$ is a $\mathbb{C}$-linear semisimple abelian category, with finitely many simple objects. This fact already has lots of consequences which we will use repeatedly. For instance, every simple module has $\mathrm{Hom}(\mathcal{S}_i, \mathcal{S}_i) = \mathbb{C}$, and every module $\mathcal{R}$ has a canonical decomposition isomorphism

$$\mathcal{R} \to \bigoplus_i V_{\mathcal{R}}^i \otimes \mathcal{S}_i$$

where the $V_{\mathcal{R}}^i$ are finite-dimensional vector spaces (given by $V_{\mathcal{R}}^i = \mathrm{Hom}(\mathcal{S}_i, \mathcal{R})$.)

Concretely the condition $\lambda \cdot \theta^\vee \leq k$ says that all the eigenvalues of $\theta^\vee \in \mathfrak{h}$ acting on M_λ are $\leq k$. To explain why we need this condition, recall that $(E_{-\theta})_{-1}^{k+1}|\Omega\rangle = 0$ in $L_k(\mathfrak{g})$. Acting on the primary part $M_\lambda \subset \mathcal{R}(\lambda)$, the zero mode of $(E_{-\theta})_{-1}^{k+1}|\Omega\rangle$ acts by $(E_{-\theta})_0^{k+1}$. It follows that $(E_{-\theta})_0^{k+1}$ must act by zero on M_λ. But now consider the $\mathfrak{sl}_2$ spanned by

$$E = E_\theta, \quad F = E_{-\theta}, \quad H = [E_\theta, E_{-\theta}] = \theta^\vee$$

(For $\mathfrak{g} = \mathfrak{sl}_N$ we would have $E_\theta = E_{1N}$ and $E_{-\theta} = E_{N1}$, i.e., the $\mathfrak{sl}_2$ we are talking about is just sitting at the outermost corners of our $\mathfrak{sl}_N$ matrices.) Then the largest H-eigenvalue occurring in M_λ is $\lambda \cdot \theta^\vee$, and using $\mathfrak{sl}_2$ representation theory we get that $F^{k+1} = 0$ on M_λ iff $\lambda \cdot \theta^\vee \leq k$.

Example 4.3. If $\mathfrak{g} = \mathfrak{sl}_2$ the root lattice is spanned by a single root α. The coroot $\theta^\vee = \alpha^\vee = H$ has $\theta^\vee \cdot \alpha = 2$. The weight lattice is spanned by $\omega = \frac{1}{2}\alpha$. A dominant weight $\lambda = n\omega$ thus has $\lambda \cdot \theta^\vee = n$. So the condition $\lambda \cdot \theta^\vee \leq k$ restricts us to the $k+1$ possible highest weights

$$\lambda \in \{0, \omega, \ldots, k\omega\}$$

corresponding to the irreducible $\mathfrak{sl}_2$-representations of dimensions $1, 2, \ldots, k+1$.

4.2.2. *Tensor products.* What happens when we have two $\mathcal{V}$-modules $\mathcal{R}_1, \mathcal{R}_2$ and bring them close together?

Optimistically we might hope that there is a notion of tensor product of $\mathcal{V}$-modules, with

$$\mathcal{S}_i \boxtimes \mathcal{S}_j \simeq \bigoplus_k N_{ij}^k \mathcal{S}_k \tag{4.1}$$

where the *fusion coefficient* $N_{ij}^k \in \mathbb{Z}_{\geq 0}$ is the dimension of the space of conformal blocks on $\mathbb{CP}^1$ with $\mathcal{S}_i, \mathcal{S}_j, \mathcal{S}_k^*$ inserted. (Note this dimension is independent of *where* the three points are.)

To start thinking about this, consider:

Example 4.4. Suppose $\mathfrak{g} = \mathfrak{sl}_2$. Then we have simple modules $\mathcal{S}_n$ corresponding to weights $n\omega$. Consider a conformal block Φ on $\mathbb{CP}^1$ with modules $\mathcal{S}_a, \mathcal{S}_b, \mathcal{S}_c$ inserted. We already know from our discussion of $\mathcal{V}^k(\mathfrak{g})$-blocks that Φ is determined by an intertwiner $\Phi_0 \in \mathrm{Hom}_{\mathfrak{g}}(M_a \otimes M_b \otimes M_c, \mathbf{triv})$. Recalling that $M_n = M_n^*$, such an intertwiner exists whenever M_c appears in $M_a \otimes M_b$, i.e. whenever $a + b + c \in 2\mathbb{Z}$ and $|a - b| \leq c \leq a + b$. But since we consider $L_k(\mathfrak{g})$ instead of $V^k(\mathfrak{g})$ we have an additional constraint now, coming from $(E_\theta)_{-1}^{k+1}|\Omega\rangle = 0$. This constraint turns out to impose the condition that $a + b + c \leq 2k$, known as the *depth rule*, first derived in [14].

This answer is a little weird. It means in particular that $\mathcal{R}(M_a) \boxtimes \mathcal{R}(M_b)$ cannot be $\mathcal{R}(M_a \otimes M_b)$.

Example 4.5. Suppose $\mathfrak{g} = \mathfrak{sl}_2$ and $k = 3$. Then we have simple modules $\mathcal{S}_0, \ldots, \mathcal{S}_3$, and the depth rule gives the full product table:

$\boxtimes$	$\mathcal{S}_0$	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$
$\mathcal{S}_0$	$\mathcal{S}_0$	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$
$\mathcal{S}_1$	$\mathcal{S}_1$	$\mathcal{S}_0 \oplus \mathcal{S}_2$	$\mathcal{S}_1 \oplus \mathcal{S}_3$	$\mathcal{S}_2$
$\mathcal{S}_2$	$\mathcal{S}_2$	$\mathcal{S}_1 \oplus \mathcal{S}_3$	$\mathcal{S}_0 \oplus \mathcal{S}_2$	$\mathcal{S}_1$
$\mathcal{S}_3$	$\mathcal{S}_3$	$\mathcal{S}_2$	$\mathcal{S}_1$	$\mathcal{S}_0$

For example, $\mathcal{S}_3 \boxtimes \mathcal{S}_3 \simeq \mathcal{S}_0$: the summand $\mathcal{S}_2$ from the naive $\mathfrak{sl}_2$ tensor product is excluded by $a+b+c \leq 2k$.

We can define the *fusion semiring* as the set of equivalence-classes of objects of $\mathcal{C}$, with addition given by $\oplus$ and product $\boxtimes$. It is commutative and associative (in this example one can check directly, e.g.

$$(\mathcal{S}_1 \boxtimes \mathcal{S}_1) \boxtimes \mathcal{S}_2 = (\mathcal{S}_0 \oplus \mathcal{S}_2) \boxtimes \mathcal{S}_2 = \mathcal{S}_2 \oplus (\mathcal{S}_0 \oplus \mathcal{S}_2) = \mathcal{S}_0 \oplus 2\mathcal{S}_2$$

while

$$\mathcal{S}_1 \boxtimes (\mathcal{S}_1 \boxtimes \mathcal{S}_2) = \mathcal{S}_1 \boxtimes (\mathcal{S}_1 \oplus \mathcal{S}_3) = (\mathcal{S}_0 \oplus \mathcal{S}_2) \oplus \mathcal{S}_2 = \mathcal{S}_0 \oplus 2\mathcal{S}_2.$$

By formally adjoining negatives of objects (Grothendieck ring) we get the *fusion ring*.

Exercise 4.1. Suppose $\mathfrak{g} = \mathfrak{sl}_2$ and $k = 3$. Using the table above, prove that the fusion ring is isomorphic to $\mathbb{Z}[x]/(x^4 - 3x^2 + 1)$.

Example 4.6. More generally, if $\mathfrak{g} = \mathfrak{sl}_2$, the fusion ring is $\mathbb{Z}[x]/U_{k+1}(\frac{x}{2})$ where U_{k+1} is a Chebyshev polynomial. (It represents the class of the $(k+1)$-dimensional irreducible representation; so we get the fusion ring by starting with the ring of finite-dimensional $\mathfrak{sl}_2$-modules and then formally "setting that representation to zero.")

We could try to produce a tensor product "by hand" by just defining $\mathcal{S}_i \boxtimes \mathcal{S}_j = \bigoplus_k \mathbb{C}^{N_{ij}^k} \otimes \mathcal{S}_k$, or more generally $\mathcal{R}_1 \boxtimes \mathcal{R}_2 = \bigoplus_k V_i^{\mathcal{R}_1} \otimes V_j^{\mathcal{R}_2} \otimes \mathbb{C}^{N_{ij}^k} \otimes \mathcal{S}_k$. This won't have the properties we usually want. For example, it isn't associative in any reasonable way, i.e. we don't have any canonical isomorphism $(\mathcal{S}_i \boxtimes \mathcal{S}_j) \boxtimes \mathcal{S}_k \simeq \mathcal{S}_i \boxtimes (\mathcal{S}_j \boxtimes \mathcal{S}_k)$. (It is true that the two sides decompose isomorphically into simple objects, but we don't have a map between them; it would mean giving a map $\bigoplus_\ell \mathbb{C}^{N_{ij}^\ell} \otimes \mathbb{C}^{N_{\ell k}^m} \simeq \bigoplus_\ell \mathbb{C}^{N_{jk}^\ell} \otimes \mathbb{C}^{N_{\ell i}^m}$.)

Nevertheless, for any modules $\mathcal{R}_1, \mathcal{R}_2$ we *can* define a product module $\mathcal{R}_1 \boxtimes \mathcal{R}_2$, for which we do have (4.1). More canonically we have a family of modules $\mathcal{R}_1 \boxtimes_{z_1,z_2} \mathcal{R}_2$ parameterized by $z_1, z_2 \in \mathbb{C}^2 \setminus \{z_1 = z_2\}$, with the property that $\mathrm{Hom}(\mathcal{R}_1 \boxtimes_{z_1,z_2} \mathcal{R}_2, W)$ is a space of 3-point conformal blocks with $\mathcal{R}_1, \mathcal{R}_2, W$ at z_1, z_2, ∞ respectively. The modules $\mathcal{R}_1 \boxtimes_{z_1,z_2} \mathcal{R}_2$ are all isomorphic (they have the same decomposition into simple objects), but not canonically so. Then we define $\mathcal{R}_1 \boxtimes \mathcal{R}_2$ by fixing (arbitrarily) $z_1 = 0$, $z_2 = 1$.

The module $\mathcal{R}_1 \boxtimes_{z_1,z_2} \mathcal{R}_2$ is not $\mathcal{R}_1 \otimes \mathcal{R}_2$ as a vector space. Nevertheless, informally we should think of $\mathcal{R}_1 \boxtimes_{z_1,z_2} \mathcal{R}_2$ as follows: it is some kind of completion of $\mathcal{R}_1 \otimes \mathcal{R}_2$, made into a $\mathcal{V}$-module by the rule that any mode $a_{(n)}$ acts by a big contour-integral encircling 0, z_1 and z_2, which by contour-deformation we can replace with small contours encircling z_1 and z_2 alone.

4.2.3. *Tensor products, continued.* To be more precise, the module $\mathcal{R}_1 \boxtimes_{z_1,z_2} \mathcal{R}_2$ is characterized by a universal property, like the usual tensor product of vector spaces. Here is our replacement for "bilinear map" in this context:

Definition 4.7. Given $\mathcal{V}$-modules $\mathcal{R}_1, \mathcal{R}_2, W$, a (z_1, z_2)-*intertwining map*[15]

$$F : \mathcal{R}_1 \otimes \mathcal{R}_2 \to \overline{W}$$

is a linear map induced by a conformal block Φ on $\mathbb{CP}^1$ with insertions $\mathcal{R}_1, \mathcal{R}_2, W^*$ at z_1, z_2, ∞, in the sense that $F(\chi_1, \chi_2) \cdot \eta = \Phi(\chi_1, \chi_2, \eta)$.

Now we can formulate the existence of tensor products, for finite enough vertex algebras:

Definition 4.8 (Rational and strongly rational vertex algebras).

- We say $\mathcal{V}$ is *rational* if there are finitely many simple ordinary modules, each ordinary module is a direct sum of simple modules, and all N_{ij}^k are finite.[16]
- Define the C_2 *subspace* of $\mathcal{V}$ by

$$C_2(\mathcal{V}) = \{a_{(-2)}b \mid a, b \in \mathcal{V}\}.$$

We say $\mathcal{V}$ is *strongly rational* if $\mathcal{V}$ is rational, $\mathcal{V} \simeq \mathcal{V}^*$, and $\dim \mathcal{V}/C_2(\mathcal{V}) < \infty$.[17]

In particular $\mathcal{V} = L_k(\mathfrak{g})$ is strongly rational, for $k \in \mathbb{N}$.

Proposition 4.9 (Existence of tensor products). Suppose $\mathcal{V}$ is rational and $\mathcal{R}_1, \mathcal{R}_2$ are ordinary modules. Then there is a $\mathcal{V}$-module $\mathcal{R}_1 \boxtimes_{z_1,z_2} \mathcal{R}_2$ with a (z_1, z_2)-intertwining map

$$\mathcal{R}_1 \otimes \mathcal{R}_2 \to \overline{\mathcal{R}_1 \boxtimes_{z_1,z_2} \mathcal{R}_2}$$

such that any other (z_1, z_2)-intertwining map $\mathcal{R}_1 \otimes \mathcal{R}_2 \to \overline{W}$ factors uniquely through this one, via a map of $\mathcal{V}$-modules, $\mathcal{R}_1 \boxtimes_{z_1,z_2} \mathcal{R}_2 \to W$.

See [15] for this definition, [16] for the construction. (It's unpleasantly technical, but the idea is to realize $\mathcal{R}_1 \boxtimes_{z_1,z_2} \mathcal{R}_2$ through how it behaves in correlation functions, i.e. define an element $\eta \in \mathcal{R}_1 \boxtimes_{z_1,z_2} \mathcal{R}_2$ as a linear map on a certain large space of "possible insertions outside the disc.")

[we should add much more discussion here about how to think about the tensor product]

One immediate consequence of this universal property is that $\mathrm{Hom}(\mathcal{R}_1 \boxtimes_{z_1,z_2} \mathcal{R}_2, W)$ can be identified as the space of conformal blocks with insertions of $\mathcal{R}_1, \mathcal{R}_2, W^*$ at z_1, z_2, ∞. Another is that $\mathcal{R} \boxtimes_{z_1,z_2} \mathcal{V} \simeq \mathcal{R}$ canonically.

4.2.4. *Braiding.* For the usual tensor product of vector spaces we have a canonical isomorphism $V_1 \otimes V_2 \simeq V_2 \otimes V_1$, and repeating it twice gives the identity on $V_1 \otimes V_2$.

Here the situation is different: we don't have any simple reason to expect $\mathcal{R}_1 \boxtimes_{z_1,z_2} \mathcal{R}_2 \simeq \mathcal{R}_2 \boxtimes_{z_1,z_2} \mathcal{R}_1$. What we have instead for free is

$$\mathcal{R}_1 \boxtimes_{z_1,z_2} \mathcal{R}_2 \simeq \mathcal{R}_2 \boxtimes_{z_2,z_1} \mathcal{R}_1 .$$

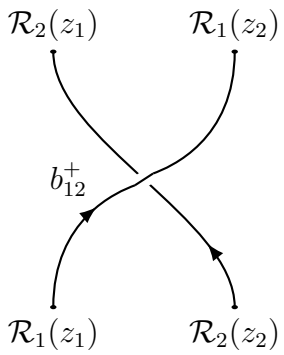
Still, for any W, using the operator $\partial_z + L_{-1}$ we can define an isomorphism

$$\mathrm{Hom}(\mathcal{R}_1 \boxtimes_{z_1,z_2} \mathcal{R}_2, W) \to \mathrm{Hom}(\mathcal{R}_1 \boxtimes_{z_1',z_2'} \mathcal{R}_2, W)$$

[15] Here $\overline{W}$ means the completion of W with respect to the grading, i.e. $\overline{W} = \prod_n W_n$.

[16] Warning: there seem to be slight variants of this definition in the literature.

[17] By the way, the space $\mathcal{V}/C_2(\mathcal{V})$ is naturally a Poisson algebra, with product $a_{(-1)}b$ and bracket $a_{(0)}b$. It's an interesting object; its spectrum is called the *associated variety* of $\mathcal{V}$. Right now though we are talking about the situation where $\mathcal{V}/C_2(\mathcal{V})$ is finite-dimensional, so its spectrum is zero-dimensional. In this case $\mathcal{V}$ is also called C_2-*cofinite* or *lisse*.



given any path from (z_1, z_2) to (z_1', z_2') which avoids the diagonal. In particular, this gives two *braiding* isomorphisms (depending which way z_1 and z_2 exchange),

$$b_{12}^{\pm}(W) : \mathrm{Hom}(\mathcal{R}_1 \boxtimes_{z_1,z_2} \mathcal{R}_2, W) \to \mathrm{Hom}(\mathcal{R}_1 \boxtimes_{z_1,z_2} \mathcal{R}_2, W)\,,$$

obeying $b_{12}^+ b_{21}^- = \mathbf{1}$.

Finally, knowing this map for every W is enough to define

$$b_{12}^{\pm} : \mathcal{R}_1 \boxtimes_{z_1,z_2} \mathcal{R}_2 \to \mathcal{R}_2 \boxtimes_{z_1,z_2} \mathcal{R}_1\,.$$

and in particular

$$b_{12}^{\pm} : \mathcal{R}_1 \boxtimes \mathcal{R}_2 \to \mathcal{R}_2 \boxtimes \mathcal{R}_1\,.$$

Sometimes we just use b for b^+.

Concretely, what is the data of b? Use the canonical decomposition $\mathcal{S}_i \boxtimes \mathcal{S}_j = \bigoplus_k V_{ij}^k \otimes \mathcal{S}_k$ (so $V_{ij}^k = V_{\mathcal{S}_i \boxtimes \mathcal{S}_j}^k$); then b amounts to a linear isomorphism

$$R_{ij}^k : V_{ij}^k \to V_{ji}^k\,.$$

It becomes a concrete matrix if we choose bases in all V_{ij}^k.

4.2.5. *Factorization.* Now we need another important analytic fact:

Proposition 4.10 (Factorization for strongly rational vertex algebras). Suppose $\mathcal{V}$ is strongly rational. Then every conformal block on the sphere with 4 ordinary modules inserted can be obtained by plumbing two blocks with 3 ordinary modules inserted, giving when z_1 is close to z_2 and z_3 is close to z_4 [draw a picture here]

$$\mathrm{Conf}(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4) = \bigoplus_i \mathrm{Conf}(\mathcal{R}_1, \mathcal{R}_2, \mathcal{S}_i) \otimes \mathrm{Conf}(\mathcal{S}_i^*, \mathcal{R}_3, \mathcal{R}_4)\,. \tag{4.2}$$

As a practical matter, part of what this means is that we can describe the behavior of correlation functions when two primary insertions $\mathcal{R}_1(z_1)$ and $\mathcal{R}_2(z_2)$ are brought close together using the L_{-1} action. We use the following:

Exercise 4.2. Let $\mathcal{V}$ be any conformal vertex algebra. Consider a conformal block on the sphere with ordinary module insertions $\mathcal{R}_{1,2,3}$ at $0, w, \infty$, with primary vectors $\chi_{1,2,3}$ of conformal weights h_i. Prove that the differential equation

$$\partial_w \langle \chi_1(0)\chi_2(w)\chi_3(\infty)\rangle = \langle \chi_1(0) L_{-1}\chi_2(w)\chi_3(\infty)\rangle$$

implies that

$$\langle \chi_1(0)\chi_2(w)\chi_3(\infty)\rangle = \alpha w^{-h_1-h_2+h_3}$$

for some $\alpha(\chi_1,\chi_2,\chi_3)\in\mathbb{C}$. (Hint: use the coinvariant constraints to relate the action of L_{-1} at w to the action of $L_{\geq 0}$ at all three marked points; then you will get a closed equation for $\langle\chi_1(0)\chi_2(w)\chi_3(\infty)\rangle$, which can be solved directly.)

Plugging this into the plumbing we see that for primary states χ_1 and χ_2, any conformal block in the summand ("channel") labeled by $\mathcal{S}_i$ in (4.2) will obey (inside correlation functions)

$$\chi_1(z_1)\chi_2(z_2) = (z_1-z_2)^{h_i-h_2-h_1}\chi_i(z_2)+\cdots$$

for some primary χ_i in $\mathcal{S}_i$, and in particular the monodromy when z_1 is taken around z_2 will be multiplication by $\mathrm{e}^{2\pi\mathrm{i}(h_i-h_2-h_1)}$.

Conversely, to prove Proposition 4.10, the main thing we need to do is to show that *all* conformal blocks have this kind of behavior as $z_1\to z_2$. I think the idea is like this. First we show that the connection on conformal blocks is *regular singular*, in the sense that the correlation functions satisfy a finite-rank system of differential equations which has only first-order poles at the diagonal. (We saw this directly for our main example (3.16). More generally it is true at least if we consider strongly rational vertex algebras and insert only ordinary modules. Note it is certainly *not* true if we consider an ordinary module approaching an "irregular module" $\mathcal{R}(u)$: then we would get a higher-order pole.) It follows then that the correlation functions have at most polynomial growth at the diagonal (but they could have logs). Next we *construct* some $\mathcal{V}$-module directly from the local behavior of the correlation functions at the diagonal. Then we use the assumed finiteness properties to prove that this $\mathcal{V}$-module is ordinary, conclude it decomposes as a direct sum of ordinary modules, and get the corresponding decomposition for the correlation function. This has been carried out in [...].

4.2.6. *Associator.* At any rate, suppose we know Proposition 4.10. Then we consider the space of 4-point blocks with $\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3,W$ inserted, at points $z_1<z_2<z_3$ in $\mathbb{R}$ and ∞. When z_2 is close to z_1 this is

$$\bigoplus_i \mathrm{Hom}(\mathcal{R}_1\boxtimes\mathcal{R}_2,\mathcal{S}_i)\otimes\mathrm{Hom}(\mathcal{S}_i\boxtimes\mathcal{R}_3,W^*)$$

In turn this is equal to[18]

$$\mathrm{Hom}((\mathcal{R}_1\boxtimes\mathcal{R}_2)\boxtimes\mathcal{R}_3,W^*)$$

Similarly when z_2 is close to z_3 we can identify the space of 4-point blocks with

$$\mathrm{Hom}(\mathcal{R}_1\boxtimes(\mathcal{R}_2\boxtimes\mathcal{R}_3),W^*)\,.$$

Continuation in the z_2 parameter then gives an isomorphism

$$\mathrm{Hom}((\mathcal{R}_1\boxtimes\mathcal{R}_2)\boxtimes\mathcal{R}_3,W)\to\mathrm{Hom}(\mathcal{R}_1\boxtimes(\mathcal{R}_2\boxtimes\mathcal{R}_3),W)\,.$$

[18] Indeed, in any finite semisimple abelian category $\mathcal{C}$, we have $\bigoplus_i \mathrm{Hom}(A,\mathcal{S}_i)\otimes F(\mathcal{S}_i)=F(A)$ for any additive functor $F:\mathcal{C}^{\mathrm{op}}\to\mathrm{Vect}$. We apply it here with $A=\mathcal{R}_1\boxtimes\mathcal{R}_2$ and $F(A)=\mathrm{Hom}(A\boxtimes\mathcal{R}_3,W)$. To think about it concretely imagine $F(A)=\mathrm{Hom}(A,B)$: then it says $\bigoplus_i\mathrm{Hom}(A,\mathcal{S}_i)\otimes\mathrm{Hom}(\mathcal{S}_i,B)=\mathrm{Hom}(A,B)$. To prove that version: we have an obvious map from the left side to the right; what we need is to invert it. For this use the canonical decomposition $A\simeq\bigoplus_j V_{j,A}\otimes\mathcal{S}_j$; that induces an isomorphism $\mathrm{Hom}(A,\mathcal{S}_i)\otimes\mathrm{Hom}(\mathcal{S}_i,B)\to\bigoplus_j V_{j,A}^*\otimes\mathrm{Hom}(\mathcal{S}_j,\mathcal{S}_i)\otimes\mathrm{Hom}(\mathcal{S}_i,B)=V_{i,A}^*\otimes\mathrm{Hom}(\mathcal{S}_i,B)=\mathrm{Hom}(V_{i,A}\otimes\mathcal{S}_i,B)$, and then taking the direct sum over i gives $\mathrm{Hom}(A,B)$ as desired.

For those who know more about category theory than I do: if I understand right, this is an example of the co-Yoneda lemma, applied to the coend $\int^{X\in\mathcal{C}}\mathrm{Hom}(A,X)\otimes F(X)$ of the functor $G(X,Y):\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathrm{Vect}$ defined by $G(X,Y)=\mathrm{Hom}(A,Y)\otimes F(X)$.

Having this for every W is equivalent to having an *associator* map

$$a_{1,2,3} : (\mathcal{R}_1 \boxtimes \mathcal{R}_2) \boxtimes \mathcal{R}_3 \to \mathcal{R}_1 \boxtimes (\mathcal{R}_2 \boxtimes \mathcal{R}_3)\,.$$

What is the concrete data here? Recall that we have the canonical multiplicity vector spaces V_{ij}^k, $\dim V_{ij}^k = N_{ij}^k$, with a canonical isomorphism

$$\mathcal{S}_i \boxtimes \mathcal{S}_j \simeq \bigoplus_k V_{ij}^k \otimes \mathcal{S}_k\,.$$

Then the associator amounts to a collection of linear isomorphisms

$$F_{ijk}^{\ell} : \bigoplus_p V_{ij}^p \otimes V_{pk}^{\ell} \to \bigoplus_q V_{iq}^{\ell} \otimes V_{jk}^q\,,$$

or more atomically

$$[F_{ijk}^{\ell}]_{pq} : V_{ij}^p \otimes V_{pk}^{\ell} \to V_{iq}^{\ell} \otimes V_{jk}^q\,.$$

They are sometimes called F*-matrices*, or $6j$ *symbols* because of the 6 indices $ijk\ell pq$. They don't literally become matrices unless we choose bases in each V_{ij}^k. They satisfy the *pentagon relation* which we will discuss below.

(I should emphasize that this is not something spooky: the same sort of objects would appear already for the category of finite-dimensional representations of a compact Lie group for instance.)

4.2.7. *Twist.* There is an operator

$$\theta_{\mathcal{R}} : \mathcal{R} \to \mathcal{R}$$

given by the action of $\exp(2\pi \mathrm{i} L_0)$. It acts by a scalar in each simple module.

4.2.8. *Double duals.* As we have remarked, we have a canonical isomorphism $\mathcal{R}^{**} \simeq \mathcal{R}$ (sometimes called *pivotal structure*).

4.2.9. *Identity object.* The object $\mathbf{1} = \mathcal{V}$ is the identity for the tensor product, in the sense that we have given isomorphisms $\lambda_X : \mathbf{1} \otimes X \to X$ and $\rho_X : X \otimes \mathbf{1} \to X$.

4.2.10. *Evaluation.* Given $\mathcal{R}$ we can build a conformal block on $\mathbb{CP}^1$ with $\mathcal{R}^*$ at z_1, $\mathcal{R}$ at z_2, and $\mathcal{V}^* = \mathcal{V}$ at ∞. Once we somehow choose a normalization of this conformal block, it gives an element in $\mathrm{Hom}(\mathcal{R}^* \boxtimes_{z_1,z_2} \mathcal{R}, \mathcal{V})$ i.e.

$$\mathrm{ev}_{\mathcal{R},z_1,z_2} : \mathcal{R}^* \boxtimes_{z_1,z_2} \mathcal{R} \to \mathcal{V}$$

In particular, this gives *two* morphisms,

$$\mathrm{ev}_{\mathcal{R}}^L : \mathcal{R}^* \boxtimes \mathcal{R} \to \mathcal{V}, \qquad \mathrm{ev}_{\mathcal{R}}^R : \mathcal{R} \boxtimes \mathcal{R}^* \to \mathcal{V}$$

We will fix the normalizations below.



4.2.11. *Coevaluation.* Now consider a conformal block on $\mathbb{CP}^1$ with $\mathcal{R}$ at z_1, $\mathcal{R}^*$ at z_2, and $\mathcal{V}^* = \mathcal{V}$ at ∞, with some choice of normalization. Such a block induces for us a map

$$\mathrm{Hom}(\mathcal{V}, \mathcal{S}) \to \mathrm{Hom}(\mathcal{R} \boxtimes_{z_1,z_2} \mathcal{R}^*, \mathcal{S})$$

for each simple $\mathcal{S}$. (Namely, if $\mathcal{S} = \mathcal{V}$ then this means a map $\mathbb{C} \to \mathrm{Hom}(\mathcal{R} \boxtimes_{z_1,z_2} \mathcal{R}^*, \mathcal{V})$, which we can get from the given conformal block; and if $\mathcal{S} \neq \mathcal{V}$ then both sides are the zero vector space.) Thus it induces a morphism

$$\mathrm{coev}_{\mathcal{R},z_1,z_2} : \mathcal{V} \to \mathcal{R} \boxtimes_{z_1,z_2} \mathcal{R}^* .$$

Again this gives two morphisms,

$$\mathrm{coev}^R_{\mathcal{R}} : \mathcal{V} \to \mathcal{R}^* \boxtimes \mathcal{R}, \qquad \mathrm{coev}^L_{\mathcal{R}} : \mathcal{V} \to \mathcal{R} \boxtimes \mathcal{R}^*$$

and again we will fix the normalizations below.



4.2.12. *Coherence.* There are many coherence relations between these various structures. For instance,

(1) All of the data are *natural* with respect to morphisms.

(2) We have the triangle identity:

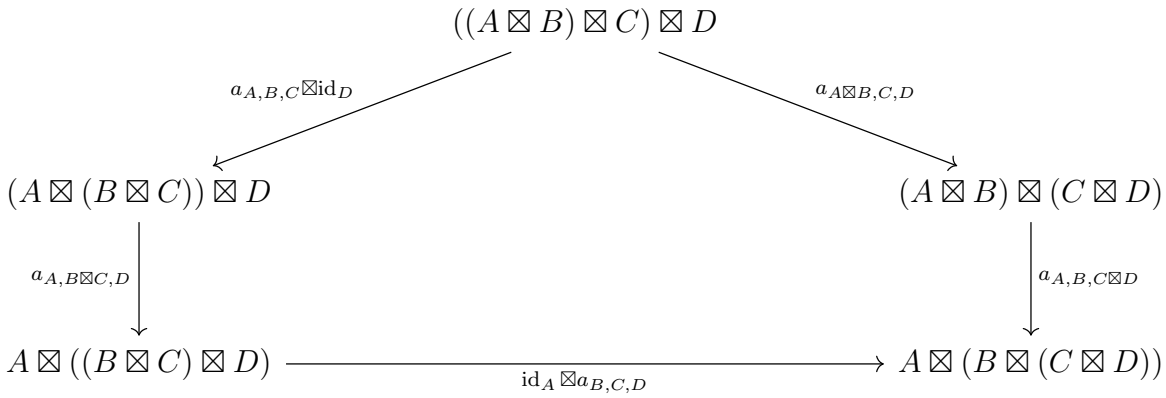
$$\begin{array}{ccc} \big((X \otimes \mathbf{1}) \otimes Y\big) & \xrightarrow{a_{X,\mathbf{1},Y}} & \big(X \otimes (\mathbf{1} \otimes Y)\big) \\ & {\scriptstyle \rho_X \otimes \mathrm{id}_Y} \searrow & \downarrow {\scriptstyle \mathrm{id}_X \otimes \lambda_Y} \\ & & \big(X \otimes Y\big) \end{array}$$

(3) The unit element is transparent to braiding:

$$b_{X,\mathbf{1}} = \lambda_X^{-1} \circ \rho_X, \quad b_{\mathbf{1},X} = \rho_X^{-1} \circ \lambda_X .$$

(4) The associator obeys a pentagon relation relating five different ways of bracketing the tensor product of four objects:

$$a_{A,B,C\boxtimes D} \circ a_{A\boxtimes B,C,D} = (\mathrm{id}_A \boxtimes a_{B,C,D}) \circ a_{A,B\boxtimes C,D} \circ (a_{A,B,C} \boxtimes \mathrm{id}_D).$$

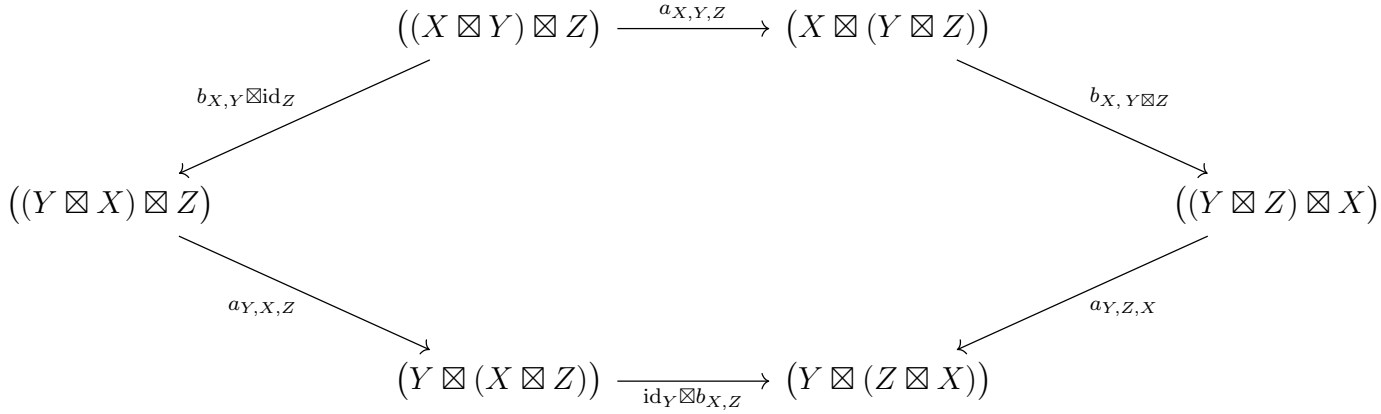


We can understand it as coming from a homotopy between paths in configuration space. The key point is that all of the tensor products we consider can be described in terms of conformal blocks where *all insertions are on the real line*, and the configuration space of ordered 4-tuples of points $t_1 < t_2 < t_3 < t_4$ on the real line is simply connected [draw figure].) For instance: the space $\mathrm{Hom}(A \boxtimes (B \boxtimes (C \boxtimes D)), W^*)$ is a space of 5-point conformal blocks constructed by iterated plumbing, which converges in a region where $t_4 - t_3 \ll t_3 - t_2 \ll t_2 - t_1$. The other bracketings are similar. Then the associator corresponds to analytic continuation of the correlation functions from one of these convergence domains to another.

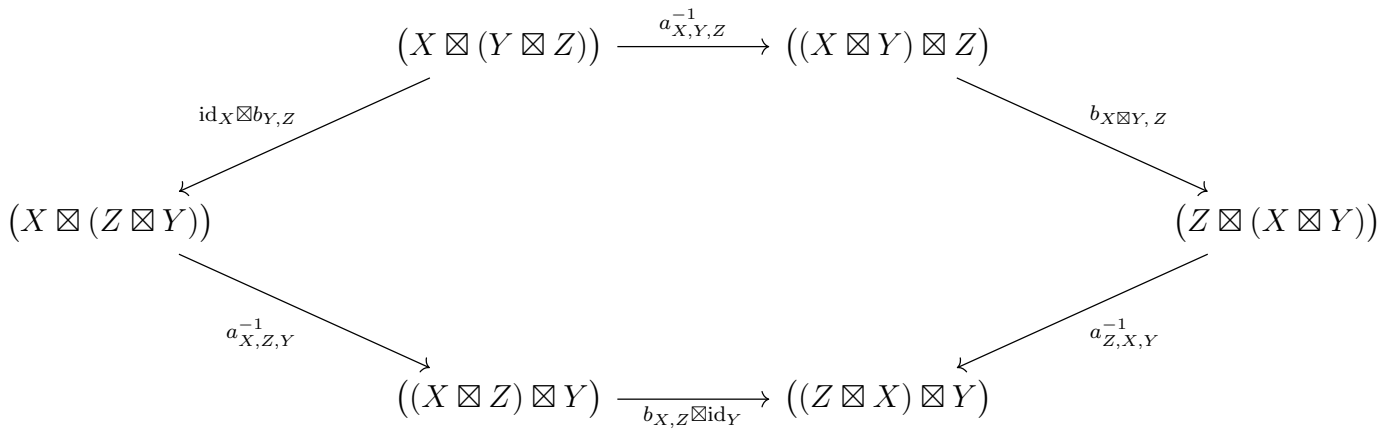
One pretty way of formulating this is: interpret factorization as saying that the bundle of conformal blocks extends over a *compactification* of the moduli of curves, to include stable nodal curves. Then the pentagon relation can be formulated as a homotopy in the moduli space $\overline{\mathcal{M}_{0,5}}(\mathbb{R})$ of real stable curves of genus zero with 5 marked points; this moduli space is a pentagon, with the five bracketings correspond to the five vertices.

(5) The braiding and associator together obey two hexagon relations. The first connects braiding X to the right across $Y \boxtimes Z$ with braiding it across Y and Z separately; the second is similar for braiding Z to the left.

$$a_{Y,Z,X} \circ b_{X,\, Y \boxtimes Z} \circ a_{X,Y,Z} = (\mathrm{id}_Y \boxtimes b_{X,Z}) \circ a_{Y,X,Z} \circ (b_{X,Y} \boxtimes \mathrm{id}_Z).$$



$$a^{-1}_{Z,X,Y} \circ b_{X \boxtimes Y,\, Z} \circ a^{-1}_{X,Y,Z} = (b_{X,Z} \boxtimes \mathrm{id}_Y) \circ a^{-1}_{X,Z,Y} \circ (\mathrm{id}_X \boxtimes b_{Y,Z}).$$



(As with the pentagon relation, these identities come from homotopies between paths in the configuration space of triples of points, where we *mostly* keep z_1, z_2, z_3 on the real line, but for the braiding we let them move slightly outside the real domain.)

(6) The coevaluation and evaluation obey zig-zag (straightening) relations, connecting coev^L and ev^L, or coev^R and ev^R:

$$\rho_X \circ (\mathrm{id}_X \boxtimes \mathrm{ev}^L) \circ a_{X,X^*,X} \circ (\mathrm{coev}^L \boxtimes \mathrm{id}_X) \circ \lambda_X^{-1} = \mathrm{id}_X, \tag{4.3}$$

$$\lambda_{X^*} \circ (\mathrm{ev}^L \boxtimes \mathrm{id}_{X^*}) \circ a^{-1}_{X^*,X,X^*} \circ (\mathrm{id}_{X^*} \boxtimes \mathrm{coev}^L) \circ \rho^{-1}_{X^*} = \mathrm{id}_{X^*}.$$

$$\lambda_X \circ (\mathrm{ev}^R \boxtimes \mathrm{id}_X) \circ a^{-1}_{X,X^*,X} \circ (\mathrm{id}_X \boxtimes \mathrm{coev}^R) \circ \rho^{-1}_X = \mathrm{id}_X,$$

$$\rho_{X^*} \circ (\mathrm{id}_{X^*} \boxtimes \mathrm{ev}^R) \circ a_{X^*,X,X^*} \circ (\mathrm{coev}^R \boxtimes \mathrm{id}_{X^*}) \circ \lambda^{-1}_{X^*} = \mathrm{id}_{X^*}. \tag{4.4}$$

We think of these relations as partially fixing the normalization of ev and coev, which we left unfixed until now.

(7) Braiding is compatible with evaluation and coevaluation:

$$\mathrm{coev}^R_X = b^{-1}_{X^*,X} \circ \mathrm{coev}^L_X\,, \qquad \mathrm{ev}^R_X = \mathrm{ev}^L_X \circ b^{-1}_{X^*,X}\,,$$

or equivalently,

$$\mathrm{coev}^L_X = b_{X^*,X} \circ \mathrm{coev}^R_X, \qquad \mathrm{ev}^L_X = \mathrm{ev}^R_X \circ b_{X^*,X}.$$

These relations mean that the structure is somewhat redundant: we could *define* coev^R in terms of coev^L and the braiding, and similarly we could define ev^R in terms of ev^L and the braiding. We can think of them as fixing the normalization of coev^R, ev^R completely in terms of those of coev^L, ev^L. They imply $\mathrm{ev}^L_X \circ \mathrm{coev}^R_X = \mathrm{ev}^R_X \circ \mathrm{coev}^L_X$.

Example 4.11. If $\mathcal{R} = \mathcal{R}(M)$, then we may choose $\mathrm{coev}^L_{\mathcal{R}}(\mathbf{1})$ to be the standard conformal block, which in particular acts on $m \in M$, $\eta \in M^*$ by $\Phi(m,\eta) = \eta \cdot m$. If we do so, then $\mathrm{coev}^R_{\mathcal{R}}(\mathbf{1})$ is fixed to [sign] $\Phi(m,\eta) = \exp(2\pi \mathrm{i} h_M)\eta \cdot m$.

Note there is a *convention* here: we could have replaced $b_{X,Y} \to b^{-1}_{Y,X}$ in these formulas. As far as I understand, we have to live with this convention, and different authors even make different choices here.

(8) The twist is compatible with the braiding and associator, in the sense that:

$$\theta_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}}\,, \quad a_{X,Y,Z} \circ \theta_{(X\boxtimes Y)\boxtimes Z} = \theta_{X\boxtimes(Y\boxtimes Z)} \circ a_{X,Y,Z}\,, \quad \theta_{X\boxtimes Y} = b_{Y,X} \circ b_{X,Y} \circ (\theta_X \boxtimes \theta_Y)\,.$$

Let us discuss how to prove the last identity here. The intuitive picture is: acting with a twist on $X \boxtimes Y$ means applying $\exp(2\pi \mathrm{i} L_0)$ to the whole composite, which amounts to applying it to each of X and Y separately, plus moving the insertion point at 1 around a circle around 0. To turn this into an actual proof, we need to think about how we even define the action of L_0 (or any vertex algebra element for that matter) on a tensor product. As usual the answer has to be in terms of the universal property: it's enough to give the action of L_0 on all conformal blocks. We use the formula $L_0 = \oint zT(z)\mathrm{d}z$, and contour deformation: this shows that with insertions at 0 and w, L_0 acts by $L_0 \otimes \mathrm{id} + \mathrm{id} \otimes L_0 + w\,\mathrm{id} \otimes L_{-1}$. The third vector field is $w\partial_w$, the generator of dilations. [...]

(9) The twist is compatible with evaluation and coevaluation, in the sense that twisting on either leg is equivalent:[19]

$$(\theta_X \boxtimes \mathrm{id}_{X^*}) \circ \mathrm{coev}_X^L = (\mathrm{id}_X \boxtimes \theta_{X^*}) \circ \mathrm{coev}_X^L, \qquad \mathrm{ev}_X^L \circ (\theta_{X^*} \boxtimes \mathrm{id}_X) = \mathrm{ev}_X^L \circ (\mathrm{id}_{X^*} \boxtimes \theta_X). \tag{4.5}$$

$$(\theta_{X^*} \boxtimes \mathrm{id}_X) \circ \mathrm{coev}_X^R = (\mathrm{id}_{X^*} \boxtimes \theta_X) \circ \mathrm{coev}_X^R, \qquad \mathrm{ev}_X^R \circ (\theta_X \boxtimes \mathrm{id}_{X^*}) = \mathrm{ev}_X^R \circ (\mathrm{id}_X \boxtimes \theta_{X^*}). \tag{4.6}$$

All of these properties together make $\mathcal{C}$ into a *premodular tensor category* or *ribbon fusion category*. I hope I have now listed all the axioms. For much more on these kinds of categories see e.g. [17, 18, 19].

One observation you quickly make when you start playing with $\mathcal{C}$ is: the objects a, λ, ρ are needed to make the equations sensible, but you can always guess where you need to put them, and so it should be possible to use a reduced notation where you simply omit them. This is indeed the case:

Theorem 4.12 (Coherence theorem). Every premodular tensor category $\mathcal{C}$ is equivalent to one which is *strict*, meaning that $X\boxtimes(Y\boxtimes Z) = (X\boxtimes Y)\boxtimes Z$ on the nose, similarly $\mathbf{1}\boxtimes X = X = X\boxtimes\mathbf{1}$, and a, λ, ρ are all identity maps.

This theorem is stated at least in [6] and the version for monoidal categories is in [19]. Here "equivalence" means it's an equivalence of linear categories, compatible in an appropriate way with the monoidal structure, braiding and twist. The upshot is that for some purposes it is safe to pretend that the category $\mathcal{C}$ is indeed strict.

4.3. **Link invariants.** Now given any premodular tensor category $\mathcal{C}$ we can consider oriented ribbon link diagrams L, where each component is labeled by an object $\mathcal{R}$ of $\mathcal{C}$ (reversing orientation swaps $\mathcal{R} \to \mathcal{R}^*$).

Such a diagram gives a morphism $Z_L : \mathbf{1} \to \mathbf{1}$ in $\mathcal{C}$: cups/caps give $\mathrm{coev}_{\mathcal{R}}$ and $\mathrm{ev}_{\mathcal{R}}$, twists/kinks give $\theta_{\mathcal{R}}^{\pm 1}$, crossings give $b^{\pm}_{\mathcal{R},\mathcal{R}'}$. We can also sprinkle in associators as necessary, or maybe better, pass to a strictification of $\mathcal{C}$, so we don't need them.

Since $\mathrm{Hom}(\mathbf{1},\mathbf{1}) = \mathbb{C}$ we can think of Z_L as a number associated to the link diagram. Now the nontrivial claim: Z_L is actually an invariant of the ribbon link, not only of the link diagram. This amounts to checking that it is invariant under various moves. All of the needed invariance follow from the coherences we have already discussed:

(1) Zig-zag identities (pulling a strand straight): use (4.3)-(4.4).

(2) Reidemeister I: see an exercise below (this is the most interesting one).

[19] This condition is sometimes written more succinctly as $\theta_{X^*} = (\theta_X)^*$,

(3) Reidemeister II: cancel $b_{X,Y}$ with $b_{X,Y}^{-1}$.

(4) Reidemeister III: see an exercise below.

(5) Pitchfork identities (pulling a strand across a cup or cap): see an exercise below.

(6) Moving twists across cups/caps: use (4.5)-(4.6).

Exercise 4.3. Prove the "pitchfork" identities:

$$(\mathrm{id}_X \boxtimes b_{Y,X^*}) \circ (b_{Y,X} \boxtimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_Y \boxtimes \mathrm{coev}_X) = \mathrm{coev}_X \boxtimes \mathrm{id}_Y,$$
$$(b_{X,Y} \boxtimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_X \boxtimes b_{X^*,Y}) \circ (\mathrm{coev}_X \boxtimes \mathrm{id}_Y) = \mathrm{id}_Y \boxtimes \mathrm{coev}_X,$$
$$(\mathrm{id}_Y \boxtimes \mathrm{ev}_X) \circ (b_{X^*,Y} \boxtimes \mathrm{id}_X) \circ (\mathrm{id}_{X^*} \boxtimes b_{X,Y}) = \mathrm{ev}_X \boxtimes \mathrm{id}_Y,$$
$$(\mathrm{ev}_X \boxtimes \mathrm{id}_Y) \circ (\mathrm{id}_{X^*} \boxtimes b_{Y,X}) \circ (b_{Y,X^*} \boxtimes \mathrm{id}_X) = \mathrm{id}_Y \boxtimes \mathrm{ev}_X\,.$$

Exercise 4.4. Prove the Reidemeister III identity, and draw the corresponding picture:

$$(b_{Y,Z} \boxtimes \mathrm{id}_X) \circ (\mathrm{id}_Y \boxtimes b_{X,Z}) \circ (b_{X,Y} \boxtimes \mathrm{id}_Z) = (\mathrm{id}_Z \boxtimes b_{X,Y}) \circ (b_{X,Z} \boxtimes \mathrm{id}_Y) \circ (\mathrm{id}_X \boxtimes b_{Y,Z}).$$

Exercise 4.5. Prove the Reidemeister I identity, and draw the corresponding picture: [check sign]

$$(\mathrm{id}_X \boxtimes \mathrm{ev}_X^L) \circ (b_{X,X} \boxtimes \mathrm{id}_X) \circ (\mathrm{id}_X \boxtimes \mathrm{coev}_X^L) = \theta_X\,.$$

[think of the ribbon structure as coming from the dependence on tangent vectors]

Exercise 4.6. Suppose $\mathcal{S}$ is a simple object with $\mathcal{S} \boxtimes \mathcal{S} \simeq \mathbf{1}$. Then prove that

$$R^{\mathbf{1}}_{\mathcal{S}\mathcal{S}} = \theta_{\mathcal{S}}\,.$$

(You could do this concretely by thinking about correlation functions on the sphere, or more abstractly using the premodular tensor category structure.)

4.4. **Examples.** Let me describe what these structures actually are in two baby examples.

Example 4.13. Take $\mathcal{V} = L_1(\mathfrak{sl}_2)$. In this case we just have simple modules $\mathcal{S}_0, \mathcal{S}_1$. The only nontrivial fusion rule is $\mathcal{S}_1 \boxtimes \mathcal{S}_1 \simeq \mathcal{S}_0$.

The only nontrivial R-matrix is $R^0_{11} = \mathrm{i}$, and the twist is $\theta_1 = \mathrm{i}$. (We can calculate the latter directly, since $\mathcal{S}_1$ has highest weight $h_1 = C_2(M_1)/2(k+h^\vee) = \frac{3}{2}/2(3) = \frac{1}{4}$; then this determines the former, using the exercise above.)

(Note it means that these particles are "semions," like half of a fermion: when you exchange them you get a factor of i.)

[should also discuss F-matrices, carefully; all scalars ± 1 but which?]

Example 4.14. Take $\mathcal{V} = L_2(\mathfrak{sl}_2)$. This case is a bit more interesting.[20]

We have three simple modules $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$, with conformal weights

$$h_0 = 0, \quad h_1 = \frac{3}{16}, \quad h_2 = \frac{1}{2}\,,$$

and corresponding twists

$$\theta_0 = 1, \quad \theta_1 = \mathrm{e}^{3\pi\mathrm{i}/8}, \quad \theta_2 = -1\,.$$

[20] The fusion rules we get here match with what physicists call "Ising anyon" statistics. But, warning: I think the R-matrix is not that of the usual Ising model.

The nontrivial fusion rules are

$$\mathcal{S}_1 \boxtimes \mathcal{S}_1 \simeq \mathcal{S}_0 \oplus \mathcal{S}_2, \quad \mathcal{S}_1 \boxtimes \mathcal{S}_2 \simeq \mathcal{S}_1, \quad \mathcal{S}_2 \boxtimes \mathcal{S}_2 \simeq \mathcal{S}_0 \,.$$

Now we describe the R and F matrices. They will depend on choices of bases in the nontrivial invariant spaces V_{ij}^k; thus we have $(\mathbb{C}^\times)^4$ worth of ambiguity.

Note that V_{ij}^k is a space of 3-point blocks on the sphere, identified with the *classical* multiplicity space $H_{ij}^k = \mathrm{Hom}_{\mathfrak{g}}(M_i \otimes M_j, M_k)$. So, concretely, we can fix all the ambiguity by choosing bases in H_{ij}^k.

The R-matrices are

$$R_{11}^0 = \mathrm{e}^{5\pi\mathrm{i}/8}, \quad R_{11}^2 = \mathrm{e}^{\pi\mathrm{i}/8}, \quad R_{12}^1 = R_{21}^1 = \mathrm{i}, \quad R_{22}^0 = -1.$$

Here $R_{11}^0 = -\exp(\pi\mathrm{i}(h_0 - 2h_1))$, $R_{11}^2 = \exp(\pi\mathrm{i}(h_2 - 2h_1))$, and $R_{22}^0 = \exp(\pi\mathrm{i}(h_0 - 2h_2))$. The exponential factor in R_{ii}^k comes from the half-monodromy of $z^{h_k - 2h_i}$. The extra $\pm$ sign is the eigenvalue of the swap acting on the *classical* multiplicity space H_{ii}^k. Finally, the choice of R_{12}^1 is a gauge choice, which then fixes R_{21}^1 since $R_{12}^1 R_{21}^1 = \exp(2\pi\mathrm{i}h_2) = -1$.

Now for the F-matrices. There are 10 scalar F-matrices, e.g.

$$F_{212}^1 : V_{21}^1 \otimes V_{12}^1 \to V_{12}^1 \otimes V_{21}^1, \qquad F_{121}^2 : V_{12}^1 \otimes V_{11}^2 \to V_{11}^2 \otimes V_{21}^1$$

both of which are determined by the hexagon relations: $F_{212}^1 = -1$ and $F_{121}^2 = \mathrm{i}R_{12}^1 = -1$. Note the first is fully gauge invariant while the second depended on a gauge choice. If we maintain this same choice, the 8 other scalar F-matrices are all $+1$.

There is one 2-dimensional fusion space

$$\mathcal{S}_1 \boxtimes \mathcal{S}_1 \boxtimes \mathcal{S}_1 \simeq \mathcal{S}_1 \oplus \mathcal{S}_1 \,.$$

It gives rise to a single F-matrix,

$$F_{111}^1 : (V_{11}^0 \otimes V_{01}^1 \oplus V_{11}^2 \otimes V_{21}^1) \to (V_{10}^1 \otimes V_{11}^0 \oplus V_{12}^1 \otimes V_{11}^2) \,.$$

Let us try to calculate F_{111}^1 directly from the vertex algebra. Actually we will do the computation a bit more generally, for any $k \geq 2$ instead of just $k = 2$. We consider conformal blocks on the sphere with 4 marked points at $0, x, 1, \infty$, all carrying the representation $\mathcal{R}(M_1)$, where we recall M_1 is the fundamental representation of $\mathfrak{sl}_2$. Now we consider varying x. The connection in the bundle of conformal blocks then takes the concrete form[21]

$$\left[\partial_x - \frac{1}{k+2}\left(\frac{T_{12}}{x} + \frac{T_{23}}{x-1}\right)\right]\psi = 0 \,, \tag{4.7}$$

acting on the classical invariant subspace $H \subset M_1^{\otimes 4}$, which has dimension 2. (Happily, this 2 matches the dimension of the space of 4-point conformal blocks, i.e. in this example there is no additional truncation to worry about.)

For x near 0 ($|x| < 1$), given a vector $\varphi \in H_{11}^a \otimes H_{a1}^1$, we can get a conformal block ψ by plumbing. What the plumbing would give us directly is a power series in x,

$$\psi = \sum_n x^n \psi_n = x^{\lambda_a}\varphi(e_i, e^i) + \cdots$$

where $\lambda_a = h_a - 2h_1$, so $\lambda_0 = -\frac{3}{2(k+2)}$ and $\lambda_2 = \frac{1}{2(k+2)}$. The higher terms are the contributions from descendants in $\mathcal{S}_a$. Happily we don't need to write out the whole power series: it is determined from φ, by the differential equation (4.7). Indeed, we could have constructed the same power

[21] More generally, with n punctures in the finite plane and one more at ∞ we would write the operator $\partial_{z_i} - \frac{1}{k+h^\vee}\sum_{j\neq i}\frac{T_{ij}}{z_i - z_j}$ – just like (3.16).

series by Frobenius's method for constructing solutions of ODE around regular singularities: every eigenvector φ of the residue matrix $\frac{1}{k+2}T_{12}$ gives a solution of the ODE. (To be precise, this is true as long as no two eigenvalues differ by an integer; that's the case here, since their difference is $\frac{2}{k+2}$.) The fusion channels $H^a_{11} \otimes H^1_{a1} \subset H$ indeed give eigenlines of T_{12}.

Now suppose x is near 1. Then given a vector $\varphi' \in H^1_{1a} \otimes H^a_{11}$ we again get a conformal block ψ' by plumbing, now of the form

$$\psi' = \sum_n (1-x)^n \psi_n = (1-x)^{\lambda_a} \varphi'(e_i, e^i) + \cdots$$

Again these blocks can be built by Frobenius's method, now starting from the eigenvector φ' of T_{13}.

Now we look at the space of solutions of (4.7) on the interval $(0,1)$. We have identified this space both with $\oplus_a H^a_{11} \otimes H^1_{a1}$ and with $\oplus_a H^1_{1a} \otimes H^a_{11}$, so we get a map between these two: that is the F-matrix F^1_{111}.

To go further we should fix our bases. Let's choose generators as follows. Define M_1 to be the standard representation, and $M_2 = \operatorname{Sym}^2 M_1$. Let $\epsilon : M_1 \otimes M_1 \to \mathbb{C}$ be the standard antisymmetric pairing.

(1) For H^0_{11} use the antisymmetric pairing ϵ.

(2) For H^2_{11} take the projection to the symmetric part.

(3) For H^1_{01} and H^1_{10} use the identity map.

(4) For H^1_{21} use the intertwiner

$$m((uv) \otimes w) = \frac{1}{\sqrt{3}}(\epsilon(v,w)u + \epsilon(u,w)v)\,.$$

(5) For H^1_{12} use the intertwiner

$$m(u \otimes (vw)) = \frac{1}{\sqrt{3}}(\epsilon(u,v)w + \epsilon(u,w)v)\,.$$

Once we have made these choices, everything else is fixed. The $\phi_a : M_1^3 \to M_a$ are

$$\phi_0(u,v,w) = \epsilon(u,v)w, \quad \phi_2(u,v,w) = \frac{1}{\sqrt{3}}(\epsilon(v,w)u + \epsilon(u,w)v)\,,$$

and similarly $\phi'_a : M_1^3 \to M_a$ are

$$\phi'_0(u,v,w) = \epsilon(v,w)u, \quad \phi'_2(u,v,w) = \frac{1}{\sqrt{3}}(\epsilon(u,v)w + \epsilon(u,w)v)\,.$$

The change-of-basis between these two is

$$F_{cl} = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\,, \tag{4.8}$$

and relative to the basis (ϕ_0, ϕ_2) we have

$$T_{12} = \begin{pmatrix} -\frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad T_{23} = F_{cl}^{-1} T_{12} F_{cl}\,.$$

Then (4.7) becomes a fully explicit 2×2 matrix equation in the plane, with regular singularities at 0, 1 and ∞. Such equations can always be reduced to hypergeometric equations, and for

those there is a connection formula available which relates the monodromy eigenbasis at 0 to the monodromy eigenbasis at 1. That's what we need here. To write the result, let

$$q = \exp(\pi \mathrm{i}/(k+2)), \qquad [n]_q = q^{n-1} + q^{n-3} + \cdots + q^{-n+1} = \frac{\sin(\frac{n\pi}{k+2})}{\sin(\frac{\pi}{k+2})}\,.$$

Then (after some work, omitted here!) the connection formula gives

$$F^1_{111} = \frac{1}{[2]_q}\begin{pmatrix} -1 & \sqrt{[3]_q} \\ \sqrt{[3]_q} & 1 \end{pmatrix}.$$

As $k \to \infty$ this recovers the classical change-of-basis (as it must since in that limit the equation just becomes $\partial_z \psi = 0$.)

Exercise 4.7. Verify (4.8) by direct calculation.

4.5. **Unitarity.** In the case $\mathcal{V} = L_k(\mathfrak{g})$, $k \in \mathbb{N}$, we have even more structure. Suppose we equip $\mathfrak{g}$ with the compact involution $\sigma(X) = -X^*$, and define $\sigma : V^k(\mathfrak{g}) \to V^k(\mathfrak{g})$ by $\sigma(|\Omega\rangle) = |\Omega\rangle$ and $\sigma(J^X) = J^{\sigma(X)}$. Then:

Proposition 4.15. All of the ordinary modules for $L_k(\mathfrak{g})$ are unitary.

This extends the fact that the finite-dimensional modules for $\mathfrak{g}$ itself are unitary. Moreover it implies a kind of unitarity for the category $\mathcal{C}$, as follows. [...]

4.6. **Quantum dimensions and quantum traces.** Now how to calculate the link invariants in practice? One simple example of such a link invariant would be the case of the unknot. The invariant for an unknot labeled with X is called the *quantum dimension* d_X. It has

$$d_X = \mathrm{ev}^R_X \circ \mathrm{coev}^L_X = \mathrm{ev}^L_X \circ \mathrm{coev}^R_X\,.$$

There is a direct way of computing the quantum dimension if we already know the F-matrices:

Proposition 4.16. Choose bases for $V^{\mathbf{1}}_{XX^*} = \mathrm{Hom}(\mathbf{1}, X \boxtimes X^*)$ and $V^{\mathbf{1}}_{X^*X} = \mathrm{Hom}(\mathbf{1}, X^* \boxtimes X)$ given by $\mathrm{coev}^L_X(\mathbf{1})$ and $\mathrm{coev}^R_X(\mathbf{1})$ respectively. Then we have

$$d_X = 1/(F^X_{XX^*X})_{\mathbf{11}}$$

Proof. Call $(X \boxtimes X^*) \boxtimes X$ and $X \boxtimes (X^* \boxtimes X)$ respectively the t and s channels. Let ψ^μ_t and ψ^μ_s be the t- and s-channel bases of $\mathrm{Hom}(X \boxtimes X^* \boxtimes X, X)$ induced by the chosen bases of the multiplicity spaces. We have

$$\psi^{\mathbf{1}}_t = \frac{1}{d_X}(\mathrm{ev}^R_X \boxtimes \mathrm{id}_X), \qquad \psi^{\mathbf{1}}_s = \frac{1}{d_X}(\mathrm{id}_X \boxtimes \mathrm{ev}^L_X).$$

By definition of the F-matrix,

$$\psi^{\mathbf{1}}_t = \sum_\mu (F^X_{XX^*X})_{\mu\mathbf{1}}\, \psi^\mu_s.$$

Now let

$$e = \frac{1}{d_X}\,\mathrm{coev}^R_X \circ \mathrm{ev}^L_X \in \mathrm{End}(X^* \boxtimes X).$$

We have $e^2 = e$, so e is the projector onto the $\mathbf{1}$-summand in $X^* \boxtimes X$. Therefore

$$\psi_s^\mu \circ (\mathrm{id}_X \boxtimes e) = \begin{cases} \psi_s^{\mathbf{1}}, & \mu = \mathbf{1}, \\ 0, & \mu \neq \mathbf{1}. \end{cases}$$

Thus

$$\psi_t^{\mathbf{1}} \circ (\mathrm{id}_X \boxtimes e) = (F^X_{XX^*X})_{\mathbf{11}}\, \psi_s^{\mathbf{1}}.$$

On the other hand,

$$\begin{aligned} \psi_t^{\mathbf{1}} \circ (\mathrm{id}_X \boxtimes e) &= \frac{1}{d_X^2} (\mathrm{ev}_X^R \boxtimes \mathrm{id}_X) \circ (\mathrm{id}_X \boxtimes \mathrm{coev}_X^R \circ \mathrm{ev}_X^L) \\ &= \frac{1}{d_X^2} (\mathrm{id}_X \boxtimes \mathrm{ev}_X^L) = \frac{1}{d_X}\, \psi_s^{\mathbf{1}}, \end{aligned}$$

using the zig-zag identity. Comparing the two expressions, we obtain

$$(F^X_{XX^*X})_{\mathbf{11}} = \frac{1}{d_X},$$

as desired. □

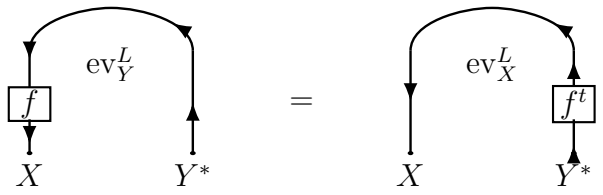
There is a little subtlety about self-dual objects:

Definition 4.17. The *transpose* of an element $f \in \mathrm{Hom}(X, Y)$ is an element $f^t \in \mathrm{Hom}(Y^*, X^*)$ defined by

$$f^t = (\mathrm{ev}_Y^R \boxtimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_{Y^*} \boxtimes f \boxtimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_{Y^*} \boxtimes \mathrm{coev}_X^L)\,.$$

The transpose obeys (and is characterized by)

$$\mathrm{ev}_Y^L \circ (f \boxtimes \mathrm{id}_{Y^*}) = \mathrm{ev}_X^L \circ (\mathrm{id}_X \boxtimes f^t), \quad \mathrm{ev}_Y^R \circ (\mathrm{id}_{Y^*} \boxtimes f) = \mathrm{ev}_X^R \circ (f^t \boxtimes \mathrm{id}_X).$$



Suppressing the pivotal identifications $X \simeq X^{**}$ and $Y \simeq Y^{**}$, we have

$$f^{tt} = f.$$

We also have the canonical isomorphism

$$\mathrm{Hom}(X, Y^*) \simeq \mathrm{Hom}(X \boxtimes Y, \mathbf{1}), \qquad f \longmapsto \mathrm{ev}_Y^L \circ (f \boxtimes \mathrm{id}_Y)$$

Composing with this, we can define the transpose of $\phi \in \mathrm{Hom}(X \boxtimes Y, \mathbf{1})$ as $\phi^t \in \mathrm{Hom}(Y \boxtimes X, \mathbf{1})$.

Definition 4.18. Suppose X is a simple object with $X \simeq X^*$. Consider the 1-dimensional vector space $\mathrm{Hom}(X, X^*) \simeq \mathrm{Hom}(X \boxtimes X, \mathbf{1})$. It has an involution $\phi \mapsto \phi^t$. This involution acts as a scalar $\nu_X \in \{\pm 1\}$, called the *Frobenius-Schur indicator* of X.

Example 4.19. The classical example arises when $\mathcal{C}$ is the category of representations of a group, say a finite group or a compact Lie group. Then, for simple X which is self-dual, there is a map of representations

$$X \otimes X \to \mathbb{C}$$

(invariant bilinear pairing), unique up to scalar multiple, which could be either symmetric or antisymmetric. We have $\nu_X = +1$ if it is symmetric, -1 if antisymmetric.

Example 4.20. For $L_k(\mathfrak{g})$ the Frobenius-Schur indicator is inherited from the finite-dimensional modules. In particular, for $L_k(\mathfrak{sl}_2)$ every simple object $\mathcal{S}_k$ is self-dual, and $\nu_k = (-1)^k$.

Exercise 4.8. Prove that $\nu_X = \theta_X R^{\mathbf{1}}_{XX}$. [check sign]

Proposition 4.21. Suppose X is simple and self-dual. Then [need to explain the basis choices here to make this equation sensible]

$$(F^X_{XXX})_{\mathbf{11}} = \nu_X \, (F^X_{XX^*X})_{\mathbf{11}} \, .$$

Proof. Choose a basis vector

$$q \in V^{\mathbf{1}}_{XX} = \mathrm{Hom}(\mathbf{1}, X \boxtimes X),$$

which we will use for the vacuum channel in both copies of $V^{\mathbf{1}}_{XX}$ in the definition of F^X_{XXX}. Let

$$p \in \mathrm{Hom}(X \boxtimes X, \mathbf{1})$$

be the dual basis vector, so that $p \circ q = \mathrm{id}_{\mathbf{1}}$. p determines a map $u : X \to X^*$, by

$$p = \mathrm{ev}^L_X \circ (u \boxtimes \mathrm{id}_X),$$

which then has

$$p^t = \mathrm{ev}^R_X \circ (\mathrm{id}_X \boxtimes u).$$

Let

$$T : \mathrm{Hom}(X \boxtimes X^* \boxtimes X, X) \longrightarrow \mathrm{Hom}(X \boxtimes X \boxtimes X, X)$$

be the isomorphism given by precomposition with $d_X \, \mathrm{id}_X \boxtimes u \boxtimes \mathrm{id}_X$. Applying T to the vacuum basis vectors we used for $F^X_{XX^*X}$ gives

$$T\left(\frac{1}{d_X} \, \mathrm{ev}^R_X \boxtimes \mathrm{id}_X\right) = p^t \boxtimes \mathrm{id}_X, \qquad T\left(\frac{1}{d_X} \, \mathrm{id}_X \boxtimes \mathrm{ev}^L_X\right) = \mathrm{id}_X \boxtimes p \, .$$

On the other hand, the vacuum basis vectors for F^X_{XXX} determined by q are

$$p \boxtimes \mathrm{id}_X, \qquad \mathrm{id}_X \boxtimes p.$$

Thus the change-of-basis coefficients for the two bases differ by ν_X, giving the desired result. □

Corollary 4.22. Suppose $X \simeq X^*$. Choose a basis for $V^{\mathbf{1}}_{XX}$ once and for all. Then we have

$$d_X = \nu_X / (F^X_{XXX})_{\mathbf{11}} \, .$$

Example 4.23. Applying this formula to $L_k(\mathfrak{sl}_2)$ we get

$$d_1 = [2]_q = q + q^{-1} \, . \tag{4.9}$$

For $k = 2$ this is $d_1 = \sqrt{2}$ as we expected. Note that d_1 matches with the Jones polynomial of the unknot, in blackboard framing.

It is interesting to explore a little more of the structure of quantum dimensions:

Exercise 4.9. Show that the quantum dimensions obey the relations

$$d_{X\boxtimes Y} = d_X d_Y, \quad d_{X\oplus Y} = d_X + d_Y\,.$$

How to calculate the quantum dimensions? For any object X consider the matrix N_X representing fusion with simple objects, $(N_X)_{jk} = \dim V^k_{X\otimes\mathcal{S}_j}$. Then the vector (d_i) is an eigenvector of the matrix N_X, with eigenvalue d_X. In general this isn't quite enough to determine (d_i). But in the unitary situation it is, because of the following:

Proposition 4.24. If $\mathcal{C}$ is unitary, then all $d_i > 0$.

Example 4.25. In [refer to the correct example] we have

$$N_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Diagonalizing these matrices we see that the only positive eigenvalues are $(d_0, d_1, d_2) = (1, \sqrt{2}, 1)$; and sure enough, this vector is a simultaneous eigenvector of all three matrices.

More generally we also have the quantum trace: given $f : X \to X$ we define

$$\operatorname{Tr} f = \operatorname{ev}^R_X \circ (f \otimes \mathrm{id}_{X^*}) \circ \operatorname{coev}^L_X = \operatorname{ev}^L_X \circ (\mathrm{id}_{X^*} \otimes f) \circ \operatorname{coev}^R_X\,.$$

4.7. **Skein relations.** Consider the case of $L_k(\mathfrak{sl}_2)$. Let $X = S_1$. We are interested in links colored by the object X. We would like to relate overcrossings to undercrossings, i.e. $b_{X,X}$ to $b^{-1}_{X,X}$. We have, with $Y = S_2$,

$$X \boxtimes X = \mathbf{1} \oplus Y$$

and the braiding $b_{X,X}$ is diagonal with respect to this decomposition, with eigenvalues[22]

$$\lambda_{\mathbf{1}} = -\exp(-\pi\mathrm{i}(2h_X)) = -q^{-3/2}, \qquad \lambda_Y = \exp(-\pi\mathrm{i}(2h_X - h_Y)) = q^{1/2}\,.$$

Similarly we can write the action of $b^{-1}_{X,X}$, and the action of $\mathrm{id}_{X\boxtimes X}$. Since these are 3 operators acting on a 2-dimensional space they must obey a linear relation, and we work it out to

$$q^{1/2} b_{X,X} - q^{-1/2} b^{-1}_{X,X} = (q - q^{-1})\,\mathrm{id}\,. \tag{4.10}$$

This is an example of a *skein relation.* [fig]

Using (4.10) and (4.9), together with the ribbon isotopy invariance, we can evaluate the invariants Z_L for oriented link diagrams L colored by the representation X. (The idea is that we can use the skein relation to replace overcrossings by undercrossings and vice versa, modulo links with fewer crossings; but by changing crossings appropriately we can always reduce the knot to a bunch of unknots, so we can evaluate it.) The resulting Z_L is an invariant of the ribbon link. We can modify it slightly to get an invariant of the link alone:

Proposition 4.26 ($L_k(\mathfrak{sl}_2)$ invariants vs Jones polynomials). Let $J(L)$ denote the Jones polynomial of the link L. Then $Z_L = q^{3w(L)/2} J(L)$, where $w(L)$ is the *writhe* of L, i.e. the number of overcrossings minus undercrossings.

[22] By $q^{1/2}$ we mean specifically $\exp(\pi\mathrm{i}/(2(k+2)))$. Some authors would call this A.

A side remark, for those who know: Z_L is very similar to the Kauffman bracket invariant. This is a little puzzling, since the Kauffman bracket invariant is defined by an *unoriented* skein relation, one which you can't really even write down if you use oriented links. You might think that it is possible to use unoriented links, since $X \simeq X^*$. But the fact that $\nu_X = -1$ makes this tricky: you have to insert "coupons" on the links representing the isomorphism $X \simeq X^*$, and you can get a sign when you slide a coupon past a coev or ev. There is a fix: you can modify the category $\mathcal{C}$, changing the twist $\theta_X \to \nu_X \theta_X$; after so doing you will have $\nu_X = 1$ for all simple objects. In this modified category you can really use unoriented links without fear, and then you get the Kauffman bracket relation on the nose. See the notorious [20], and [21].

4.8. **Verlinde loop operators.** Above we discussed colored oriented ribbon links $L \subset \mathbb{R}^3$.

More generally by the same construction, a colored oriented ribbon link in $C \times I$ produces an operator on the space $\mathrm{Conf}(C)$ of conformal blocks. In the case of a simple loop in C, this is sometimes called a *Verlinde loop operator*. The Verlinde loop operators generate a *skein algebra* which we call $\mathrm{Sk}(\mathcal{V}, C)$. In the case $\mathcal{V} = L_k(\mathfrak{sl}_2)$ we wrote explicitly the relations defining the skein algebra above. So each space $\mathrm{Conf}(C)$ is naturally a representation of $\mathrm{Sk}(\mathcal{V}, C)$.

4.9. **Irregular Verlinde operators.** When we insert irregular modules on C, there is an additional possibility: we can define *irregular Verlinde operators*. I don't know how to set up the theory generally, but let's discuss a specific example.

We revisit the setup of Example 3.32. So, suppose $\mathcal{V} = V^k(\mathfrak{g})$. We take $C = \mathbb{CP}^1$, with marked points at z_1, z_2, and ∞. At z_1 and z_2 we insert parabolic Verma modules $\mathcal{R}(M_1)$ and $\mathcal{R}(M_2)$ while at ∞ we insert the irregular module $\mathcal{R}(u)$, for some $u \in \mathfrak{g}$ (identified with $\mathfrak{g}^*$ using the invariant pairing κ.) Again, the subspace $M_1 \otimes M_2 \otimes \mathbb{C}|\Omega\rangle_u \subset \mathcal{R}(M_1) \otimes \mathcal{R}(M_2) \otimes \mathcal{R}(u)$ represents $\mathrm{Coinv}(C)$ for every z_1, z_2. The computation we did there is now modified by an extra term: we get

$$L_{-1}^{(1)} = -\frac{1}{k+h^\vee}\left(\sum_{a=1}^{\dim \mathfrak{g}} \frac{J_0^a \otimes J_0^a \otimes \mathbf{1}}{z_1 - z_2} - u \otimes \mathbf{1} \otimes \mathbf{1}\right)$$

Now fix $z_1 = z$, $z_2 = 0$. Then we get the ODE

$$\partial_z + \frac{1}{\hbar}\left(u \otimes \mathbf{1} + \frac{T}{z}\right) \tag{4.11}$$

acting on $\mathrm{Coinv}(C) \simeq M_1 \otimes M_2$, where we introduced the notation $\hbar = k + h^\vee$.

This ODE has a regular singularity at $z = 0$ as we have discussed before, but also an irregular singularity at $z = \infty$. How can we think about its solutions near the irregular singularity? A crude first approximation would be to forget the T/z term. In that approximation we could write solutions in the form

$$\psi = \exp(-uz/\hbar)\, v \otimes w \tag{4.12}$$

Now following the spirit of Frobenius's method one might try to build *actual* solutions which behave "like" (4.12) as $z \to \infty$, by some kind of series expansion around $z = \infty$. If you try this, you discover a problem: the equation actually does not have solutions which behave this way, uniformly in z. The best we can do is to pick some *sector* and ask for these asymptotics as $z \to \infty$ in that sector.

Here is a baby example of the same phenomenon. We consider the Bessel equation

$$z^2 y'' + z y' + (z^2 - \nu^2) y = 0\,.$$

Formally expanding around ∞ we expect solutions of the form[23]

$$y^R_\pm(z) \sim z^{-1/2} e^{\pm \mathrm{i} z}\Big(1 + O(z^{-1})\Big) \qquad (z \to +\infty),$$

and

$$y^L_\pm(z) \sim (-z)^{-1/2} e^{\pm \mathrm{i} z}\Big(1 + O(z^{-1})\Big) \qquad (z \to -\infty).$$

These series are not convergent. Nevertheless, there exist unique solutions (y^R_+, y^R_-) (resp. (y^L_+, y^L_-)) which have the asymptotics (4.9) as $z \to \infty$ (resp. (4.9) as $z \to -\infty$). The Stokes phenomenon is the fact that these solutions do *not* analytically continue to one another. Continuing in the upper half-plane gives

$$\begin{pmatrix} y^L_+ \\ y^L_- \end{pmatrix} = S^+ \begin{pmatrix} y^R_+ \\ y^R_- \end{pmatrix}, \qquad S^+ = \begin{pmatrix} -\mathrm{i} & 2\cos(\pi\nu) \\ 0 & -\mathrm{i} \end{pmatrix},$$

while continuing in the lower half-plane gives

$$\begin{pmatrix} y^R_+ \\ y^R_- \end{pmatrix} = S^- \begin{pmatrix} y^L_+ \\ y^L_- \end{pmatrix}, \qquad S^- = \begin{pmatrix} -\mathrm{i} & 0 \\ 2\cos(\pi\nu) & -\mathrm{i} \end{pmatrix}.$$

The diagonal entries here reflect the *formal* monodromy, i.e. the naive analytic continuation we would expect just by looking at the formal series solution. The product S^+S^- gives the actual monodromy.

We meet a similar situation for our equation (4.11), but more complicated because here there are blocks of solutions with the same exponential coefficient.

To be concrete, let us specialize to the situation where $\mathfrak{g} = \mathfrak{sl}_N$, and $u = \mathrm{i}\,\mathrm{diag}(u_1, \dots, u_N)$ with all u_i real and $u_1 < u_2 < \cdots < u_N$. Further specialize so that M_1 is the fundamental representation and write $M = M_2$. Then, for each $m \in M$ we have N distinguished solutions $\psi^{L,R}_i(m)$, which have roughly

$$\psi^{L,R}_i(m, z) \sim \exp(-\mathrm{i} u_i z/\hbar)\, e_i \otimes m$$

as $z \to \pm\infty$. Now we have again Stokes phenomenon, now with the Stokes matrices $S^\pm$ organized into $N \times N$ block matrices, whose entries are valued in $\mathrm{End}(M)$. The remarkable fact[24] proven in [22] is that these Stokes matrix entries give a representation of the quantum group $U_q(\mathfrak{gl}_N)$ on M.

For instance consider the case $\mathfrak{g} = \mathfrak{sl}_2$. Then we have

$$S^+ = \begin{pmatrix} k & e \\ 0 & k^{-1} \end{pmatrix}, \qquad S^- = \begin{pmatrix} k & 0 \\ f & k^{-1} \end{pmatrix}$$

[23] More exactly

$$y^R_\pm(z) = z^{-1/2}\, e^{\pm \mathrm{i} z} \sum_{k \geq 0} a^{(\pm)}_k\, z^{-k}, \qquad a^{(\pm)}_0 = 1,$$

with

$$a^{(\pm)}_k = \frac{(\pm\mathrm{i})^k}{k!\, 8^k} \prod_{m=1}^{k} \Big(4\nu^2 - (2m-1)^2\Big).$$

So

$$y^R_\pm(z) = z^{-1/2} e^{\pm \mathrm{i} z}\left(1 \pm \frac{\mathrm{i}(4\nu^2-1)}{8z} - \frac{(4\nu^2-1)(4\nu^2-9)}{2!\,(8z)^2} \mp \frac{\mathrm{i}(4\nu^2-1)(4\nu^2-9)(4\nu^2-25)}{3!\,(8z)^3} + O(z^{-4})\right).$$

[24] which I learned from Xiaomeng Xu and Yan Zhou

for some operators $e, f, k \in \mathrm{End}(M)$. Then define

$$K = k, \quad E = \frac{K^{-1}e}{q^{-1} - q}, \quad F = \frac{fK}{q - q^{-1}}\,.$$

These operators obey the relations of $U_q(\mathfrak{sl}_2)$:

$$KEK^{-1} = q^2 E, \qquad KFK^{-1} = q^{-2}F, \qquad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}\,.$$

[check this precise formula; prove it directly using skein relations]

This is a theorem about Stokes matrices, which can be formulated and proved without mentioning conformal blocks. But we think of it as an analogue of the notion of Verlinde operator. We start with a conformal block. Then we create a new insertion of the module $\mathcal{R}(M_1)$ near the irregular module at $z = +\infty$. There seem to be N different "channels" in which to create this insertion, corresponding to the N solutions ψ_i^L above. Then we use the usual connection on conformal blocks to continue this insertion to $z = -\infty$ and fuse it to the irregular module again, choosing again one of N possible channels. In this way we get operators $S_{ij}^{\pm}$ on the space of conformal blocks, giving the quantum group action.

This construction should work much more generally: if we have a Riemann surface with 3 such irregular insertions we should get 3 actions of the quantum group, etc.

4.10. **Modular structure.** So far we have discussed a lot of structure on the category $\mathcal{C}$ that comes from studying conformal blocks on the sphere. We actually get a bit more by considering the torus.

As we already discussed, by plumbing a sphere to itself we get a conformal block Φ_i on the torus $C_q = \mathbb{C}^\times/(z \sim qz)$ for each simple module $\mathcal{S}_i$, and the Φ_i span the space $V = \mathrm{Conf}(C_q)$.

Now, recall that the mapping class group of the torus acts on V. (We remarked in [...] that the action of the mapping class group is in general only projective; but the case of the torus is an exception, because we have a global section of the bundle of complex projective structures.) So we get a representation of $\mathrm{SL}(2, \mathbb{Z})$ on V. In particular the standard S generator acts; let S_{ij} denote its matrix elements, i.e.

$$S(\Phi_i) = \sum_j S_{ij}\Phi_j$$

Note that S^2 gives the action of $-1 \in \mathrm{SL}(2, \mathbb{Z})$, which is

$$S^2\Phi_i = \Phi_{i^*}\,.$$

implying

$$(S^{-1})_{ij} = S_{i^*j} = S_{ij^*}\,.$$

The torus C_q created by plumbing comes with distinguished A and B cycles: the A cycle comes from the unit circle in the original sphere, the B cycle comes from the positive real line (when q is real). For each of these cycles and each simple module $\mathcal{S}_i$ we get a Verlinde loop operator; so we have operators A_i, B_i acting on V. The action of A_i is particularly simple: it acts already *before* the plumbing, so it must be diagonal in the basis of the Φ_j,

$$A_i\Phi_j = \lambda_{ij}\Phi_j\,.$$

Equivalently, the basis $\Psi_j = S\Phi_j$ diagonalizes the B_i. On the other hand, B_i creates a copy of $\mathcal{S}_j$ and moves it close to $z = 0$, and moves $\mathcal{S}_j^*$ close to $z = \infty$, where they can fuse with the $\mathcal{S}_i$ and

$\mathcal{S}_i^*$ already there: in other words B_i acts by fusion with Φ_i, i.e.

$$B_i \Phi_j = N_{ij}^k \Phi_k \,.$$

Thus S diagonalizes the fusion operation. From here, it is a matter of pure linear algebra to derive:

Exercise 4.10. Prove that $\lambda_{ij} = \frac{S_{ij}}{S_{0j}}$. (Hint: use the fact that $B_i \Phi_0 = \Phi_i$ for all i.)

Exercise 4.11. Prove the *Verlinde formula*:

$$N_{ij}^k = \sum_a \frac{S_{ia} S_{ja} S_{k^* a}}{S_{0a}} \,.$$

5. Chern-Simons theory

5.1. **The Chern-Simons action.** Fix a compact, simple, simply connected Lie group G and level $k \in \mathbb{Z}$. Also fix a connection $\mathrm{d} + A$, with $A \in \Omega^1(\mathfrak{g})$, on a (trivialized)[25] principal G-bundle over a closed oriented 3-manifold M. Then the *Chern–Simons action* is an invariant of the connection, valued in $\mathbb{R}/2\pi\mathbb{Z}$. For $G = \mathfrak{sl}(N)$ we can write it as[26]

$$S_{\mathrm{CS}}(A) = \frac{k}{4\pi} \int_M \mathrm{Tr} \left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right) .$$

Exercise 5.1. Prove that $S_{\mathrm{CS}}(A)$ is an invariant of the connection up to equivalence. Concretely, this means that $S_{\mathrm{CS}}(A)$ is invariant under gauge transformations $A \to A^g = g^{-1} A g + g^{-1} \mathrm{d} g$, where $g : M \to G$. Note that this requires $k \in \mathbb{Z}$. (Hint: first prove that S_{CS} is invariant under infinitesimal gauge transformations $\delta A = \mathrm{d}X + [A, X]$, $X \in \Omega^1(\mathfrak{g})$; then prove that for finite gauge transformations $S_{\mathrm{CS}}(A^g) - S_{\mathrm{CS}}(A) = -\frac{k}{12\pi} \int_M g^* \omega$ where $\omega \in \Omega^3(G)$ is given by $\omega = \mathrm{Tr}((g^{-1}\mathrm{d}g)^3)$. Now, $H^3(G, \mathbb{Z}) = \pi_3(G) = \mathbb{Z}$, generated by the image of the de Rham class of ω, and then what remains is to figure out the correct normalization. It is probably good to do this all for $G = SU(2)$ first, where we know a concrete isomorphism $G \simeq S^3$.)

Exercise 5.2. Prove that the critical points of $S_{\mathrm{CS}}(A)$ are *flat* connections, i.e. connections for which $F_A = \mathrm{d}A + A \wedge A = 0$.

We can also discuss the case $G = U(1)$, but this isn't simply connected, and there is a problem: one cannot always trivialize G-bundles over 3-manifolds. So, one needs some more machinery. One low-tech way is the following. For $G = U(1)$, given a G-bundle P with connection over M, one can always find a 4-manifold X with $\partial X = M$, and such that P extends to a bundle Q with connection over X. Then we define

$$\mathrm{CS}(P) = \frac{k}{4\pi} \int_X F_Q \wedge F_Q \,.$$

[25] Fortunately, when G is compact, simple, simply connected, we have $\pi_k(G) = 0$ for $k \leq 2$, and it follows that every principal G-bundle over a 3-manifold is trivializable.

[26] For $\mathfrak{g} = \mathfrak{sl}(N)$ the form $A \wedge A \wedge A$ means in components $\frac{1}{6} \epsilon_{ijk} (A_i A_j A_k) \mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k$. For general $\mathfrak{g}$ we could write the cubic term in S_{CS} as $\sum_{a,b,c=1}^{\dim \mathfrak{g}} \kappa(T_a, [T_b, T_c]) A^a \wedge A^b \wedge A^c$.

Note that if P is trivializable then we can take Q to be trivial over X, and then write the connection in Q as $\mathrm{d} + A$ for $A \in \Omega^1(X)$; then

$$\mathrm{CS}(P) = \frac{k}{4\pi}\int_X F_Q \wedge F_Q = \frac{k}{4\pi}\int_X \mathrm{d}(A \wedge \mathrm{d}A) = \frac{k}{4\pi}\int_M A \wedge \mathrm{d}A$$

so this agrees with the definition we gave before.

For more general G (not necessarily connected, and components not necessarily simply connected) one can give a more abstract definition of S_{CS}. It was originally given in [23], further developed in [24]. The generalization of the "level k" is a class $k \in H^4(BG, \mathbb{Z})$.

5.2. **The partition function.** At any rate, once we have defined S_{CS}, we can *formally* write the *Chern-Simons partition function*: it is

$$Z_{\mathrm{CS}}(M) = \int \mathcal{D}A\, e^{\mathrm{i}S_{\mathrm{CS}}(A)}\,.$$

Here the integral is supposed to run over connections *modulo gauge transformations*, to the invariance under gauge transformations is important. Witten [25] introduced this quantum field theory and gave a method of computing the partition function, which was more fully and rigorously developed in many subsequent works, particularly [26].

There is a pesky detail we should discuss. In the most direct approach to calculating the partition function, one needs to introduce a gauge-fixing, and this involves a Riemannian metric g on M. Formally one would expect the partition function to be independent of g, but actually this is not quite right: there is an *anomaly*, as follows. If g_0 and g_1 are two Riemannian metrics on M, then fix a metric g on $\widetilde{M} = M \times [0,1]$, restricting to g_0 and g_1 at the two ends. Let $R \in \Omega^2(\widetilde{M}, \mathrm{End}(T\widetilde{M}))$ be the curvature of g. Then we have

$$\frac{Z_{\mathrm{CS}}(M, g_1)}{Z_{\mathrm{CS}}(M, g_0)} = \exp\left[-\frac{2\pi\mathrm{i}\,c}{24}\int_{M\times[0,1]} p_1(R)\right]\,.$$

where p_1 denotes the first Pontryagin form,

$$p_1(R) = -\frac{1}{8\pi^2}\,\mathrm{Tr}(R \wedge R)\,.$$

Exercise 5.3. Prove that p_1 is conformally invariant.

Thus for closed M, $Z_{\mathrm{CS}}(M, g)$ is "almost topological:" it depends on the metric, but in a simple and controllable way.

Two remarks:

- In the literature on TFTs, e.g. [26], we will find a slightly different version of the story, where we require M to carry some additional topological structure s such as *framing* (trivialization of TM) or *2-framing* or p_1 *structure*, and we produce $Z'_{\mathrm{CS}}(M, s)$ which depends on s in a controlled way, e.g. changing a framing by 1 unit (twisting the framing by the generator of $\pi_3(SO(3)) \simeq \mathbb{Z}$) changes the invariant by $\mathrm{e}^{2\pi\mathrm{i}c/6}$. The relation between the two stories is that once given this data one can define a *gravitational Chern-Simons invariant* $Z_{\mathrm{grav}}(M, g, s)$ which depends on both, such that $Z_{\mathrm{CS}}(M, g) = Z'_{\mathrm{CS}}(M, s)Z_{\mathrm{grav}}(M, g, s)$.

5.3. **The topological field theory.** Now, what do we expect when we allow M to be a compact 3-manifold with boundary? On general QFT grounds, one expects:[27]

- For every oriented compact surface Y with a Riemannian metric there is a finite-dimensional vector space $V(Y)$.
- Suppose M is an oriented Riemannian 3-manifold, with boundary Y equipped with the induced boundary orientation. Then the path integral of Chern-Simons theory provides a vector $Z_{CS}(M) \in V(Y)$.
- Disjoint union: $V(Y_1 \sqcup Y_2) = V(Y_1) \otimes V(Y_2)$.
- Orientation reversal: $V(-Y) = V(Y)^*$.
- Gluing: if N is obtained by gluing of M along a boundary component Y, then $Z_{\mathrm{CS}(N)} = \mathrm{ev}_Y(Z_{\mathrm{CS}(M)})$

5.4. **The relation to conformal blocks.** Witten made the following remarkable proposal: *For $k > 0$, $V(Y)$ is the space of conformal blocks of $L_k(\mathfrak{g})$ on Y.* (In particular $V(Y)$ only depends on the conformal class of the metric, not the full metric.)

5.5. **Canonical quantization.** How would a physicist usually describe $V(Y)$? The standard answer in quantum mechanics or quantum field theory is that one should perform *canonical quantization* of the space of critical points of the action on $Y \times \mathbb{R}$. Indeed this space carries a canonical symplectic structure.

[include a basic explanation of how you compute this symplectic structure]

In our case, a critical point means a flat connection on $Y \times \mathbb{R}$. There is a finite-dimensional space $\mathcal{M} = \mathcal{M}^{flat}(C, G)$ of these (not quite a manifold, but at least a stack, which is a manifold away from some singular points). It gets a symplectic form which can be written $\Omega = k\omega_{AB}$, where

$$\omega_{AB}(a_1, a_2) = \frac{1}{4\pi} \int_C \mathrm{Tr}(a_1 \wedge a_2)$$

where a_1, a_2 belong to $T_{(E,\nabla)}\mathcal{M}^{flat}(C, G) = \{a \in \mathrm{End}\, E \mid \mathrm{d}_\nabla a = 0\}$.

To carry out the quantization of a symplectic manifold $\mathcal{M}$, one standard device is *geometric quantization*. This requires us to choose a *polarization*, and we will choose a complex one: this means we have to find a complex structure on $\mathcal{M}$, and then find a holomorphic line bundle $\mathcal{L} \to \mathcal{M}$ with a Hermitian connection, whose curvature is $F = \mathrm{i}k\omega_{AB}$. The output of geometric quantization then is the vector space of holomorphic sections, $V = H^0(\mathcal{L})$. [watch out for metaplectic correction!] One could also ask which operators act on this space, quantizing the smooth functions on $\mathcal{M}$; here the first answer is somewhat disappointing: geometric quantization only guarantees to quantize Hamiltonians H generating (real parts of) holomorphic automorphisms, and there will be very few of these [maybe we should think instead about Berezin-Toeplitz?] Nevertheless, we will see that the space does carry a lot of interesting structure.

Let us now specialize to $G = SU(n)$ to be most concrete. For the particular $\mathcal{M}$ we are considering, we can get a complex structure on $\mathcal{M}$ by choosing a complex structure on C itself:

Definition 5.1 (Moduli of holomorphic bundles). Let $\mathrm{Bun}_{G_\mathbb{C}}(C)$ be the moduli space parameterizing holomorphic $G_\mathbb{C}$-bundles over C.

$\mathrm{Bun}_{G_\mathbb{C}}(C)$ is a smooth algebraic stack. It has an associated coarse moduli space of semistable bundles, which is an algebraic variety; we'll also call that $\mathrm{Bun}_{G_\mathbb{C}}(C)$ (sorry). This coarse moduli

[27] This is a slight modification of Atiyah's axioms for topological field theory, to include some mild metric dependence.

space has a (Zariski) open subset parameterizing *stable* holomorphic $G_{\mathbb{C}}$-bundles; that subset is a complex manifold. [add refs]

Theorem 5.2 (Narasimhan-Seshadri). We have an isomorphism of coarse moduli spaces

$$\mathcal{M}^{flat}(C, G) \simeq \mathrm{Bun}_{G_{\mathbb{C}}}(C)\,.$$

This isomorphism maps the locus of stable bundles to the locus of irreducible connections.

Definition 5.3 (Theta line bundle). Pick a line bundle κ over C, with $\deg \mathcal{L} = g - 1$.[28] By the Riemann-Roch formula we then have $\chi(\mathcal{E} \otimes \kappa) = 0$, for any $\mathcal{E} \in \mathrm{Bun}_{G_{\mathbb{C}}}(C)$. A generic $\mathcal{E}$ has $h^0(\mathcal{E} \otimes \kappa) = 0$; the locus

$$\Theta = \{\mathcal{E} \in \mathrm{Bun}_{G_{\mathbb{C}}}(C) \mid h^0(\mathcal{E} \otimes \kappa) > 0\} \subset \mathrm{Bun}_{G_{\mathbb{C}}}(C)$$

forms a divisor in $\mathrm{Bun}_{G_{\mathbb{C}}}(C)$. Then, $\mathcal{L}_\Theta$ is the line bundle over $\mathrm{Bun}_{G_{\mathbb{C}}}(C)$ associated to this divisor.

Note that $\mathcal{L}_\Theta$ depends on the choice of κ, though its isomorphism class doesn't. This line bundle has $c_1(\mathcal{L}_\Theta) = [\omega_{AB}]$, so it admits a connection whose curvature is ω_{AB}. Then, the output of geometric quantization in our setup is $V = H^0(\mathcal{L}_\Theta^k)$. This is sometimes called the space of *nonabelian theta functions*. (In the case $G = U(1)$ they are ordinary theta functions.)

5.6. **The abelian case.** In the abelian case all these ingredients can be made fully explicit.

Proposition 5.4 (The $U(1)$ case and the Jacobian). Let C be a compact connected Riemann surface of genus g.

(1) For a holomorphic line bundle $L \to C$, its degree is

$$\deg L = \int_C c_1(L) = \frac{\mathrm{i}}{2\pi} \int_C F_{\nabla_h},$$

where ∇_h is the Chern connection of any Hermitian metric h on L.

(2) Smooth complex line bundles on C are classified by degree:

$$\{\text{isomorphism classes of smooth complex line bundles}\} \;\simeq\; H^2(C, \mathbb{Z}) \;\simeq\; \mathbb{Z}.$$

(3) If L is a fixed smooth line bundle of degree 0, the moduli space of flat unitary connections on L modulo unitary gauge is canonically

$$\mathcal{M}^{flat}_{U(1)}(C) \;\simeq\; H^1(C, \mathbb{R})/H^1(C, \mathbb{Z}) \;\simeq\; U(1)^{2g}.$$

(4) The holonomy (Riemann–Hilbert) map

$$\mathrm{RH} : \mathcal{M}^{flat}_{U(1)}(C) \longrightarrow \mathrm{Hom}(\pi_1(C), U(1))$$

is an isomorphism.

(5) (Narasimhan–Seshadri in rank 1) The map sending a flat unitary connection ∇ to the holomorphic bundle $(L, \bar\partial_\nabla)$ identifies

$$\mathcal{M}^{flat}_{U(1)}(C) \;\simeq\; \mathrm{Pic}^0(C) = \mathrm{Jac}(C).$$

[28] For instance, we could choose a spin structure on C, i.e. choose κ such that $\kappa \otimes \kappa \simeq K_C$.

Sketch. (1) The formula for $\deg L$ is Chern–Weil. On a curve, $c_1(L)$ is represented by $(\mathrm{i}/2\pi)F_{\nabla_h}$.

(2) Topological complex line bundles are classified by $c_1 \in H^2(C,\mathbb{Z})$. Since $H^2(C,\mathbb{Z}) \cong \mathbb{Z}$, the degree is a complete topological invariant.

(3) Fix one unitary connection ∇_0 on L (not necessarily flat). Any other unitary connection is $\nabla_0 + a$ with $a \in \Omega^1(C;\mathrm{i}\mathbb{R})$. Flatness for $\nabla = \nabla_0 + a$ is $F_{\nabla_0} + \mathrm{d}a = 0$. When $\deg L = 0$, there exist flat solutions; choosing one such ∇_f, all flat unitary connections are $\nabla_f + a$ with $\mathrm{d}a = 0$. Unitary gauge transformations $g : C \to U(1)$ act by $a \mapsto a - g^{-1}\mathrm{d}g$, so we quotient the space of closed imaginary-valued 1-forms by the lattice of forms for which all periods are integral. This gives $\mathcal{M}^{flat}(C) = H^1(C,\mathbb{R})/H^1(C,\mathbb{Z})$.

(4) A flat connection has path-independent parallel transport up to homotopy, hence a character of $\pi_1(C)$. Conversely, a character $\rho : \pi_1(C) \to U(1)$ defines $(\widetilde{C}\times\mathbb{C})/\pi_1(C)$ with action $\gamma\cdot(x,z) = (\gamma x, \rho(\gamma)z)$ and its descended flat connection. These constructions are inverse.

(5) If ∇ is flat unitary, then $(L,\bar\partial_\nabla)$ is holomorphic and

$$\deg L = \frac{\mathrm{i}}{2\pi}\int_C F_\nabla = 0.$$

So we get a map $\mathcal{M}^{flat}(C) \to \mathrm{Pic}^0(C)$. To see this map is surjective, suppose given $(L,\bar\partial)$ of degree 0, and choose any Hermitian metric h in L. Its Chern connection ∇_h has curvature F_h of type $(1,1)$ and integral 0. Since $c_1(L) = 0$ in de Rham cohomology, F_h is exact. Now comes the key analytic input, a little piece of Hodge theory: the $\partial\bar\partial$-lemma implies

$$F_h = \partial\bar\partial\varphi$$

for some real function φ. Rescaling the metric by $h' = \mathrm{e}^{-\varphi}h$ gives $F_{h'} = F_h - \partial\bar\partial\varphi = 0$. Hence h' has flat Chern connection, giving the desired preimage of $(L,\bar\partial)$. Injectivity follows because two flat unitary connections defining the same holomorphic structure differ by a unitary gauge transformation. □

5.7. **Relation to conformal blocks.** Above I tried to convince you that $H^0(\mathcal{L}_\Theta^k)$ has some reasonable claim to be the space of states of Chern-Simons theory on C. On the other hand, one has the following:

Theorem 5.5. Suppose $G = SU(n)$. Then the space $H^0(\mathcal{L}_\Theta^k)$ is canonically isomorphic to the space of conformal blocks on C for $L_k(\mathfrak{g})$.

This identification is due to Beauville and Laszlo [27].

[analogue for $U(1)$: $H^0(\mathcal{L}_\Theta^k)$ is space of free fermion blocks] [free fermion vertex superalgebra] The vertex superalgebra FF is defined by a simple operator product:

$$\psi_+(z)\psi_-(w) = \frac{1}{z-w} + \mathrm{reg}$$

[it contains the Heisenberg vertex algebra] [definition of conformal blocks, depending on extra datum of a spin structure] [description of free fermion blocks via Szego kernel and determinant formulas] [twisted free fermion blocks] So the twisted free fermion blocks form a line bundle over $\mathrm{Jac}\,C$. Moreover this line bundle carries a natural holomorphic structure. Its dual has a canonical section, which maps a block to $\langle 1\rangle$. [...]

5.8. **Chiral boundary conditions.** [...]

6. The center at the critical level

[we get an object $T(z)$ which transforms as an oper]

7. Some other things

7.1. **Holomorphic factorization.** This is a short answer to the question "why did people think about conformal blocks in the first place?"

In a 2-dimensional conformal field theory one computes correlation functions of observables. These correlation functions are almost factorized into products of "holomorphic" and "antiholomorphic" parts, but not exactly factorized; the nontrivial space of conformal blocks is the obstruction to factorization.

For example, in the CFT of the *free boson* Φ, one is interested in correlation functions of the form

$$\langle\!\langle J(z_1)\cdots J(z_n)\overline{J}(w_1)\cdots\overline{J}(w_m)\rangle\!\rangle$$

where classically $J=\partial\Phi$ and $\overline{J}=\overline{\partial}\Phi$. Then one finds an approximate factorization

$$\langle\!\langle J(z_1)\cdots J(z_n)\overline{J}(w_1)\cdots\overline{J}(w_m)\rangle\!\rangle = \int \mathrm{d}a\, \langle J(z_1)\cdots J(z_n)\rangle_{\Psi_a}\langle\overline{J}(w_1)\cdots\overline{J}(w_m)\rangle_{\overline{\Psi}_a}$$

where $\langle J(z_1)\cdots J(z_n)\rangle_{\Psi_a}$ are holomorphic in a strong sense — they are holomorphic in the z_i, in a, and in the moduli of C. They arise as part of the data of a conformal block $\Psi_a\in\mathrm{Conf}(C,\mathrm{Heis})$. So the conformal blocks Ψ_a are the basic building-blocks one uses to build the correlation functions of the CFT.

For the simplest example of this, suppose C is the flat torus with modulus τ, $C=\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)$, and any area A. The simplest correlation function of the free boson is the "0-point function" $\langle\!\langle 1\rangle\!\rangle$, which turns out to be given by (after zeta-function regularization and deleting the zero mode, see e.g. [28])[29]

$$\langle\!\langle 1\rangle\!\rangle = \frac{\sqrt{A}}{\sqrt{\det\Delta'}} = \frac{1}{\sqrt{\operatorname{Im}\tau}|\eta(\tau)|^2} \tag{7.1}$$

Here $\eta(\tau)$ is the Dedekind eta-function,

$$\eta(\tau)=q^{1/24}\prod_{n\geq 1}(1-q^n),\qquad q=\mathrm{e}^{2\pi\mathrm{i}\tau}\,.$$

The answer (7.1) is *almost* factorized into holomorphic and antiholomorphic, but not quite, because of the factor $\sqrt{\operatorname{Im}\tau}$. The explanation of this in terms of conformal blocks is: the Heisenberg algebra on C has conformal blocks Ψ_a parameterized by $a\in\mathbb{C}$, with the correlation function

$$\langle 1\rangle_{\Psi_a}=\frac{1}{\eta(\tau)}\exp\left(\frac{1}{4\pi\mathrm{i}}a^2\tau\right)$$

The blocks relevant for the free boson are the ones with $a\in\mathrm{i}\mathbb{R}$. Then we have

$$\langle\!\langle 1\rangle\!\rangle = \int_{a\in\mathrm{i}\mathbb{R}}\left(\frac{\mathrm{d}a}{2\pi\mathrm{i}}\right)|\langle 1\rangle_{\Psi_a}|^2 = \frac{1}{\sqrt{2\operatorname{Im}\tau}|\eta(\tau)|^2} \tag{7.2}$$

[29] In a CFT of central charge c, the partition function on a surface of Euler characteristic χ scales as $A^{\frac{c}{12}\chi}$ under overall rescaling of the metric. Here $\chi=0$ so we should have no A dependence in the end; but $\det\Delta'$ scales as A, because $\zeta_\Delta(0)=\frac{1}{6}\chi-1$, the -1 coming from the zero mode; this requirement tells us that we should compensate the removal of the zero mode by inserting the factor $\sqrt{A}$ in the numerator. We can probably get the same factor from proper treatment of the shift symmetry $\Phi\to\Phi+\alpha$?

There is a similar, but more elaborate, story for higher genus surfaces [28].

7.2. **Liouville theory.** A more sophisticated example is provided by another 2d CFT, *Liouville theory*. In this case the relevant vertex algebra is Vir^c rather than Heis.

We will need a bit of notation. First, introduce a parameter b related to the central charge by

$$c = 1 + 6Q^2, \qquad Q = b + \frac{1}{b}$$

As usual we want to consider conformal blocks with $\mathcal{V}$-modules inserted. For $\mathcal{V} = \mathrm{Vir}^c$ the basic ordinary module is a Verma module $\mathcal{R}_h$, defined by induction from a single highest-weight state $|\Omega_h\rangle$ obeying

$$L_n|\Omega_h\rangle = 0\,(n > 0), \qquad L_0|\Omega_h\rangle = h|\Omega_h\rangle\,.$$

Exercise 7.1. Take $C = \mathbb{CP}^1$ with three marked points, carrying ordinary modules $\mathcal{R}_h$. Prove that the space of conformal blocks is 1-dimensional, independent of the h_i.

Exercise 7.2. Prove that $\mathcal{R}_h^* \simeq \mathcal{R}_h$.

Combining these two exercises, we see that the structure of tensor products for Vir^c is very different from the rational examples we have discussed: it is some kind of direct integral rather than direct sum, like

$$\mathcal{R}_{h_1} \boxtimes \mathcal{R}_{h_2} = \int \mathrm{d}h\, \mathcal{R}_h$$

Now, for a simple example, consider $C_q = \mathbb{CP}^1$ with 4 punctures at $z_i = 0, q, 1, \infty$, with Verma modules $\mathcal{R}_{h_i}$ placed at the punctures. In this case we can attempt to construct blocks $\Psi_p \in \mathrm{Conf}(C_q, \mathrm{Vir}^c)$ by plumbing. More precisely I should say we can *attempt* to construct them by plumbing; the plumbing formula [...] produces an infinite series, and it is not at all clear whether it will converge. Remarkably this problem was settled recently by the probabilists [...].

Now introduce a parameter p related to h by

$$h = j(Q - j)\,, j = \frac{1}{2}Q + \mathrm{i}p$$

The full Liouville partition function can be expressed in terms of these conformal blocks. Namely we have a factorization formula similar to (7.2):[30]

$$\langle\langle 1 \rangle\rangle = \int_{p \in \mathbb{R}} (\rho(p)\,\mathrm{d}p)\, |\langle 1 \rangle_{\Psi_p}|^2$$

where we have the "Plancherel measure"

$$\rho(p) = 4 \sinh(2\pi b p) \sinh(2\pi b^{-1} p)\,,$$

and similar formulas for correlation functions. [try to be more explicit, write DOZZ etc, understand better the issues about absorbing structure constant]

[30] This is not the most common way of writing the factorization formula, but it can be found e.g. in [29].

7.3. **Tau functions.** Let $\mathcal{V}$ be the Virasoro algebra with central charge $c = 1$, $\mathcal{V} = \mathrm{Vir}_{c=1}$. Then $\mathrm{Conf}(C, \mathcal{V})$ is a module over a commutative algebra $\mathcal{A}$, the algebra of *Verlinde loop operators*. The joint eigenvectors of $\mathcal{A}$ are particularly interesting.

For instance, let C_q be $\mathbb{CP}^1$ with 4 punctures placed at points $(z_1, z_2, z_3, z_4) = (0, 1, \infty, q)$. At z_i we place a simple module for $\mathrm{Vir}_{c=1}$: a Verma module with highest weight h_i. Then consider a 1-parameter family of conformal blocks $\Psi(q) \in \mathrm{Conf}(C_q, \mathrm{Vir}_{c=1})$ which are parallel, $\nabla_{\partial_q}\Psi(q) = 0$. Suppose moreover that $\Psi(q)$ is a joint eigenvector of $\mathcal{A}$ for some q; then it is a joint eigenvector for all q.[31] Let $\tau(q)$ be the 0-point function

$$\tau(q) = \langle 1 \rangle_{\Psi(q)}$$

Then $\tau(q)$ is a τ-function of Painleve VI. This means it is a solution of an interesting differential equation [...]

7.4. **The AGT correspondence.** [...]

The AGT correspondence gives a very different interpretation of the conformal blocks which showed up in Liouville theory. We describe it in the concrete case of the 4-punctured sphere C_q which appeared above. Then

$$\langle 1 \rangle_{\Psi_a} = Z_{\mathrm{Nek}}(a, q, m, \epsilon_1, \epsilon_2)$$

The right side is the *Nekrasov instanton partition function* for $\mathcal{N} = 2$ supersymmetric Yang-Mills theory, with gauge group $G = \mathrm{SU}(2)$, and matter content given by 4 copies of the fundamental representation:

$$Z_{\mathrm{Nek}} = Z_{1-\mathrm{loop}} Z_{\mathrm{inst}}$$

Here $Z_{1-\mathrm{loop}}$ is a q-independent function

$$Z_{1-\mathrm{loop}} = \exp\left(\gamma_{\epsilon_1,\epsilon_2}(2a;\Lambda) + \gamma_{\epsilon_1,\epsilon_2}(-2a;\Lambda) + \sum_{f=1}^{4} \gamma_{\epsilon_1,\epsilon_2}(a + m_f;\Lambda) + \gamma_{\epsilon_1,\epsilon_2}(-a + m_f;\Lambda)\right),$$

in terms of a special function $\gamma_{\epsilon_1,\epsilon_2}(x;\Lambda)$ defined in [...], and Z_{inst} is an explicit series in powers of q, whose coefficients are rational functions of $a, \epsilon_1, \epsilon_2, m_f$. The surprising thing is that the series Z_{inst} also arises in a totally different way, as a generating series of integrals over the moduli space of $\mathrm{SU}(2)$-instantons in $\mathbb{R}^4$.

[give explicit formula]

8. Random matrix models

Fix a polynomial V of degree K. We consider an integral over random $N \times N$ matrices,

$$Z_N = \frac{1}{\mathrm{Vol}\ U(N)} \int \mathrm{d}M \mathrm{e}^{\mathrm{Tr}\ V(M)}$$

If K is even and V has negative leading coefficient, then we could take M to run over Hermitian matrices. In that case, we can reduce the integral to an integral over eigenvalues w_i,

$$Z_N = \frac{1}{N!} \int \left[\prod_{i=1}^{N} \frac{\mathrm{d}w_i}{2\pi \mathrm{i}}\, e^{V(w_i)}\right] \left[\prod_{i<j} (w_i - w_j)^2\right],$$

[31] The algebra $\mathcal{A}$ depends on the surface C, so here it should really be written $\mathcal{A}_q$ rather than just $\mathcal{A}$; but the dependence is locally constant as we vary the surface.

Even when V is more general, we can *define* the matrix model by an integral of this form, where the contour of integration need not be the real line: rather it is chosen to make the integral converge.

Example 8.1. If $V(w) = w^2$, then we are discussing the *Gaussian* matrix model. The partition function is $Z_N = [...]$

In addition to the partition function, we can discuss correlation functions [...]

Appendix A. Conventions on Lie algebras

For a simple Lie algebra $\mathfrak{g}$, there is a canonical invariant bilinear form κ on $\mathfrak{g}$, inducing κ^{-1} on $\mathfrak{g}^*$. We normalize it so that the long roots α of $\mathfrak{g}$ have $\kappa^{-1}(\alpha, \alpha) = 2$.

The quadratic Casimir operator is defined by

$$C_2 = \sum_i X^i X^i$$

where (X^i) is an orthonormal basis of $\mathfrak{g}$. In this basis we write

$$[X^i, X^j] = f^{ijk} X^k$$

and the structure constant f^{ijk} is antisymmetric in all exchanges of indices.

The dual Coxeter number $h^\vee$ is defined by the property that C_2 acts in the adjoint representation by the scalar $2h^\vee$. i.e. for any $Y \in \mathfrak{g}$

$$\sum_{i=1}^{\dim \mathfrak{g}} [X^i, [X^i, Y]] = 2h^\vee Y \,.$$

Taking $Y = X^j$ and summing repeated indices this is also

$$[X^i, f^{ijk} X^k] = f^{ik\ell} f^{ijk} X^\ell = 2h^\vee X^j$$

i.e.

$$f^{ik\ell} f^{ijk} = 2h^\vee \delta^{j\ell}$$

When $\mathfrak{g} = \mathfrak{sl}(N)$, the normalized form is

$$\kappa(X, Y) = \mathrm{Tr}(XY)$$

and

$$h^\vee = N \,.$$

The quadratic Casimir operator for $\mathfrak{g} = \mathfrak{sl}(2)$ is

$$C_2 = \frac{1}{2} H^2 + EF + FE \,.$$

In the representation with highest weight $n\omega$, it has eigenvalue $C_2 = \frac{1}{2} n(n+2)$. In particular, for the fundamental representation we have $C_2 = \frac{3}{2}$.

For any root $\alpha \in \mathfrak{h}^*$ the corresponding coroot $\alpha^\vee$ is determined by the condition

$$\alpha^\vee \cdot \lambda = \frac{2\kappa^{-1}(\lambda, \alpha)}{\kappa^{-1}(\alpha, \alpha)}$$

i.e.

$$\alpha^\vee = \frac{2\alpha^\sharp}{\kappa^{-1}(\alpha, \alpha)}$$

We use $\theta \in \mathfrak{h}^*$ to denote the *highest root*: it is the one maximal in the partial order (i.e. $\theta + \alpha_i$ is not a root for any simple root α_i.) For $\mathfrak{g} = \mathfrak{sl}(N)$, $\theta = \alpha_1 + \cdots + \alpha_{N-1}$, with root space spanned by E_{1N}. The corresponding coroot is $\theta^\vee = E_{11} - E_{NN} = [E_{1N}, E_{N1}]$.

A *Cartan involution* is a Lie algebra involution $\theta : \mathfrak{g} \to \mathfrak{g}$. If $\mathfrak{g}$ is complex and θ commutes with the compact involution τ, then we can define a real form of $\mathfrak{g}$ as the fixed locus of $\sigma = \theta \circ \tau$. For example, for $\mathfrak{g} = \mathfrak{sl}(N, \mathbb{C})$ we can take $\theta(X) = -X^T$, $\tau(X) = -X^*$, so $\sigma(X) = \overline{X}$, and the corresponding real form is $\mathfrak{sl}(N, \mathbb{R})$. We can also take $\theta(X) = X$, so $\sigma(X) = -X^*$, and the corresponding real form is $\mathfrak{su}(N)$.

Appendix B. The zeta-regularized determinant

In the theory of the free boson one expects that

$$\langle 1 \rangle = \sqrt{A}/\sqrt{\det{}'\Delta}$$

Since Δ acts on an infinite-dimensional space we need to explain what we mean by its determinant. Precisely we mean the *zeta-regularized* determinant.

Define

$$\zeta_\Delta(s) = \sum_{k \geq 1} \lambda_k^{-s}$$

for $\mathrm{Re}(s) > 1$, where $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ are the eigenvalues of Δ. In fact this function admits meromorphic continuation to the full s-plane. To see this consider the operator $\mathrm{e}^{-t\Delta}$ giving heat flow for time t. This is a nice integral operator with a smooth kernel $K_t(x, y)$, i.e.

$$\mathrm{e}^{-t\Delta} f = \int_{y \in C} K_t(x, y) f(y)\,.$$

As $t \to 0$ the kernel K_t concentrates around the diagonal (it converges in some distributional sense to a δ-function). On the diagonal we have[32]

$$K_t(x, x) \sim \frac{1}{4\pi t}\left(1 + \frac{R}{6}t + \cdots\right)$$

We can subtract these leading divergent terms from K_t, to get a new kernel $\widetilde{K}_t(x) = K_t(x, x) - \frac{1}{4\pi t} - \frac{R}{24\pi}$. Now, for $\mathrm{Re}(s) > 1$ we have an integral representation

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{\mathrm{d}t}{t} t^s \,\mathrm{Tr}\,{}'\mathrm{e}^{-t\Delta}$$

With the analytic continuation computed, we finally have

$$\det{}'\Delta = \exp\big(-\zeta_\Delta'(0)\big)$$

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[32] To understand the leading term $\frac{1}{4\pi t}$, note the heat kernel on $\mathbb{R}^2$ is a Gaussian, $K_t(x, y) = \frac{1}{4\pi t} \exp\left(-\frac{|x-y|^2}{4t}\right)$.

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