Problem set 4
Due Saturday September 27 at 11pm

Exercise 4.1 (10 points). For each of the following, determine whether X with the distance function d is a metric space, and prove your answer.

(1) $X = \mathbb{R}$, $d(x,y) = |x^2 - y^2|$

(2) $X = \mathbb{R}$, $d(x,y) = |x - 2y|$

(3) $X = \mathbb{R}$, $d(x,y) = \frac{|x-y|}{1+|x-y|}$

Exercise 4.2 (10 points). Let X be any set, let and $d : X \times X \to \mathbb{R}$ be the discrete metric, defined by

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

for all $x, y \in X$.

(1) Prove that, with this distance function, X is a metric space.

(2) For any $x \in X$, what is $N_\epsilon(x)$ when $\epsilon = \frac{1}{2}$, 1, and 2?

(3) Which subsets of X are open? Which are closed?

Exercise 4.3 (5 points). Show that the subset of $\mathbb{R}^2$ given by

$$E = \{(x,y) \in \mathbb{R}^2 \mid x < y\}$$

is open.

Exercise 4.4 (6 points; Rudin 2.5). Construct a bounded set of real numbers with exactly three limit points (using the standard metric on $\mathbb{R}$). (You need not prove carefully what the limit points are; it is sufficient to give the set and state what are the limit points.)

Exercise 4.5 (20 points; Rudin 2.9). Let E be a subset of a metric space. Define the *interior* of E, denoted E°, to be the set of all interior points of E.

(1) Prove that E° is always open.

(2) Prove that E is open if and only if $E^\circ = E$.

(3) Prove that, if G is an open subset of E, then $G \subset E^\circ$.

(4) Prove that the complement of E° is the closure of the complement of E.

(5) Do E and $\overline{E}$ always have the same interiors?

(6) Do E and E° always have the same closures?