Bounded-complexity approximation of filled-Julia sets

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Preliminary definitions

Notation

Iterator

$f_\kappa(z) := z^2 + \kappa$, where z, $\kappa \in \mathbb{C}$

Orbit

$\mathcal{O}(z, \kappa) := \left\{ z^{(n)} \right\}_{n=1}^{\infty}$ where $z^{(n+1)} := f_\kappa\left(z^{(n)}\right)$, $z^{(0)} = z$

Unbounded orbit

$\exists\ n_0$, such that $\left| z^{(n_0)} \right| > r(\kappa)$, where $r(\kappa) := \max\left\{ |\kappa|, 2 \right\}$

Filled Julia set

Filled Julia set

$K_\kappa := \{z : \mathcal{O}(z, \kappa) \text{ remains bounded}\}$

Unpredictability

No formula that determines based on z if orbit $\mathcal{O}(z, \kappa)$ remains bounded

Escape-time heuristic approximation of K_κ

- n_1 large integer
- If for all $n \leq n_1$, $\left|z^{(n)}\right| \leq r(\kappa)$, then assume $\mathcal{O}(z, \kappa)$ remains bounded for all $n \geq n_1$
- hence assume that $z \in K_\kappa$

This assumption maybe false

hence a picture of a fractal is only an approximation of the true filled-Julia set K_κ

Rational filled Julia set

Rational complex number

$z = \frac{p}{q} + i\left(\frac{u}{v}\right)$ where $p, q, u, v \in \mathbb{Z}$, where $q \neq 0,\ v \neq 0$

Set of rational complex numbers

$\hat{\mathbb{C}}$

Rational filled-Julia set

$\hat{K}_\kappa := K_\kappa \bigcap \hat{\mathbb{C}}$

Complexity

Complexity of integer

Encode integer m

binary string $x(m)$ of length $l(x(m)) = \lceil \log |m| \rceil + 1$ bits (1 bit for the sign)

Prefix code

binary *prefix-free* codeword $s(m) := 1^{l(x(m))} 0 x(m)$

(uniquely decodable and instantenous)

Complexity of an integer m

$\ell(m) = l(s(m)) = 2l(x(m)) + 1$, which is

$$\ell(m) = 2 \lceil \log |m| \rceil + 3$$

Complexity of rational complex number

Rational fraction, lowest term form

$\frac{p}{q} = \frac{(p,q)p'}{(p,q)q'} = \frac{p'}{q'}$ where (p,q) is the GCD of p and q

$z = \frac{p}{q} + i\left(\frac{u}{v}\right) = \frac{p'}{q'} + i\left(\frac{u'}{v'}\right)$

Encode z

by binary *prefix-free* codeword $s(z) := s(p')s(q')s(u')s(v')$

Complexity of $z = \frac{p}{q} + i\left(\frac{u}{v}\right)$

$\ell(z) = \ell(p') + \ell(q') + \ell(u') + \ell(v')$

Bounded complexity approximation of filled Julia set

$\hat{K}_\kappa^{(m)} := \left\{z \in \hat{K}_\kappa : \ell(z) \leq m\right\}$, where $\hat{K}_\kappa$ rational filled Julia set

Proposition

$\lim_{m\to\infty} \hat{K}_\kappa^{(m)} = \hat{K}_\kappa$

Oracle numbers

Oracle numbers

Kraft's inequality

binary prefix code S: $0 < \sum_{s \in S} 2^{-l(s)} \leq 1$
(special case of the LYM inequality for antichains)

Chaitin's oracle number[1]

universal Turing machine U, program p
$0 < \Omega_U := \sum_{p:\, p \text{ halts}} 2^{-l(p)} \leq 1$

Fractal oracle number[2]

$0 < \Upsilon_\kappa := \sum_{z \in \hat{\mathbb{C}} \setminus K_\kappa} 2^{-\ell(z)} \leq 1$

Fractal theory	**Theory of computation**
Rational complex number $z \in \hat{\mathbb{C}}$	Finite program p (Turing machine)
Iterator $f_\kappa(z)$ acts on z	Universal machine $U(p)$ executes p
$\ell(z)$ is description length of z	$l(p)$ is program length
Orbit $\mathcal{O}(z, \kappa)$ unbounded	Execution of p on U halts
$\Upsilon_\kappa = \sum_{z \in \hat{\mathbb{C}}: \mathcal{O}(z,\kappa) \text{ unbounded}} 2^{-\ell(z)}$	$\Omega_U := \sum_{p:\, p \text{ halts}} 2^{-l(p)}$

[1] G. Chaitin (2011)
[2] J. Ratsaby (2022)

Interesting fact about Ω_U

If the first few thousand bits of Ω_U were known

can *decide* most interesting *finitely refutable conjectures* in mathematics (find a counter-example or prove that there is none)

What about Υ_κ

If first m bits of Υ_κ were known

What could be obtained?

Proposition

Based on the first m bits of Υ_κ, Procedure `constructBCFilledJulia`(m, κ) computes $\hat{K}_\kappa^{(m)}$ in finite time

Fact 1 (consistency)

For any $z \in \hat{\mathbb{C}}$, suffices to check if $z \in \hat{K}_\kappa^{(m)}$ then $z \in K_\kappa$ (since $\hat{K}_\kappa^{(m)} \subseteq \hat{K}_\kappa \subset K_\kappa$)
This property (consistency) does not hold for escape-time approximation

Fact 2 (complete information)

This single number Υ_κ has *complete information* about the *asymptotic behavior* of the iterator f_κ, namely,

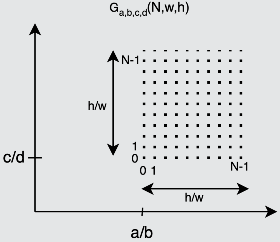
- for *any* arbitrarily large finite m
- for *infinitely* many initial points $z \in \hat{\mathbb{C}}$
- if $z \in \hat{K}_\kappa^{(m)}$ then $\mathcal{O}(z, \kappa)$ bounded

Υ_κ provides an arbitrarily-accurate approximation of the rational filled-Julia set

Results

Theorem

Theorem: Given any region of interest $G \bigcap \hat{K}_\kappa$ defined by a grid $G := G_{a,b,c,d}(N, w, h)$



It is sufficient for m to satisfy:

$$c_1 \left| \log \left(\frac{Nw}{h} \right) \right| \leq m \leq c_2 \log \left(Nwh \right)$$

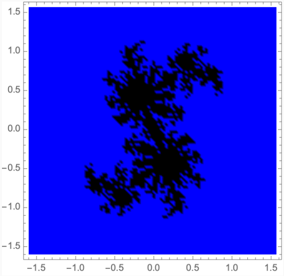
for Procedure `constructBCFilledJulia`(m, κ) to output $\hat{K}_\kappa^{(m)}$ which contains the region of interest

Examples

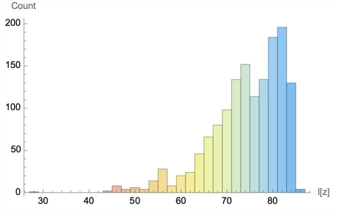
Region of interest

$G_{a,b,c,d}(N, w, h)$ the blue square

Choosing $m = 100$ yields $\hat{K}_\kappa^{(m)}$ contains the full region of interest $G \bigcap \hat{K}_\kappa$



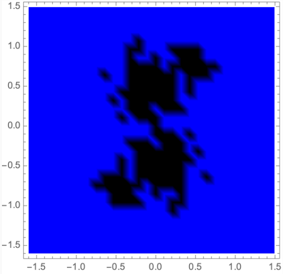
(a) Black: $G \bigcap \hat{K}_\kappa^{(m)}$



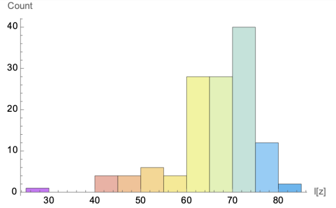
(b) Distribution of $\ell(z)$, for $z \in G \bigcap \hat{K}_\kappa$

Figure 1: $m = 100$, $N = 100$

Reduce to $N = 30$



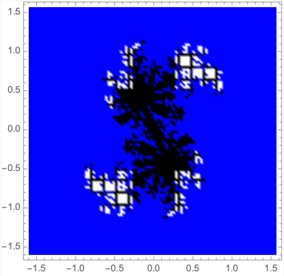
(a) Black: $G \bigcap \hat{K}_{\kappa}^{(m)}$



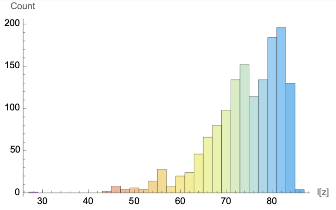
(b) Distribution of $\ell(z)$, for $z \in G \bigcap \hat{K}_{\kappa}$

Figure 2: $m = 100$, $N = 30$

Reduce to $m = 80$



(a) Black: $G \bigcap \hat{K}_\kappa^{(m)}$
White:
$\left\{z{:}z \in G \bigcap \hat{K}_\kappa, \ell(z) > m\right\}$

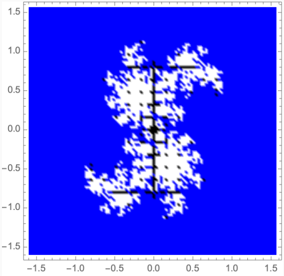


(b) Distribution of $\ell(z)$, for $z \in G \bigcap \hat{K}_\kappa$

Figure 3: $m = 80$, $N = 100$

Reduce complexity threhold m

Reduce to $m = 65$



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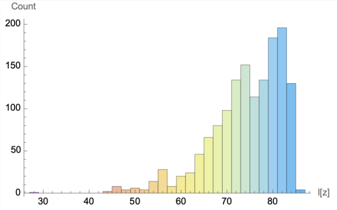
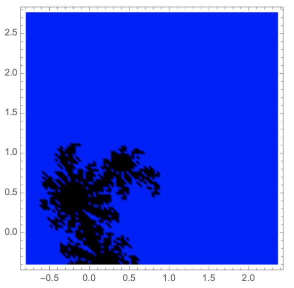


(b) Distribution of $\ell(z)$, for $z \in G \bigcap \hat{K}_{\kappa}$

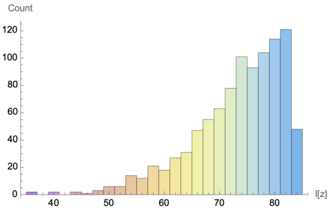
Figure 4: $m = 65$, $N = 100$

New region of interest G

New $G_{a,b,c,d}(N, w, h)$ the blue square. Fix $m = 90$, $N = 100$ side length $\frac{32}{10}$



(a) Black: $G \bigcap \hat{K}_\kappa^{(m)}$

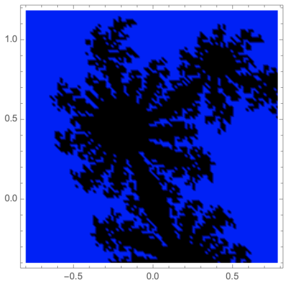


(b) Distribution of $\ell(z)$, for $z \in G \bigcap \hat{K}_\kappa$

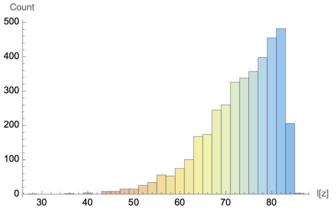
Figure 5: Side length $\frac{h}{w}$, $h = 32$, $w = 10$

Zoom in (reduce side length)

Reduce to $\frac{32}{20}$



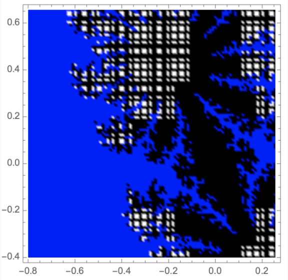
(a) Black: $G \bigcap \hat{K}_\kappa^{(m)}$



(b) Distribution of $\ell(z)$, for $z \in G \bigcap \hat{K}_\kappa$

Figure 6: Side length $\frac{h}{w}$, $h = 32$, $w = 20$

Reduce to $\frac{32}{30}$



(a) Black: $G \bigcap \hat{K}_{\kappa}^{(m)}$.
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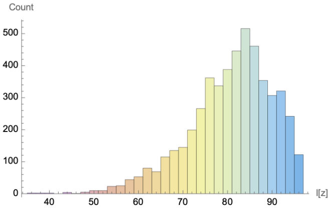
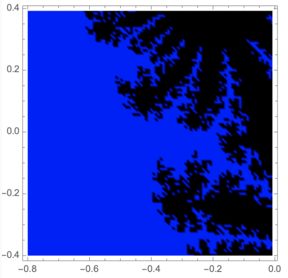


(b) Distribution of $\ell(z)$, for $z \in G \bigcap \hat{K}_{\kappa}$

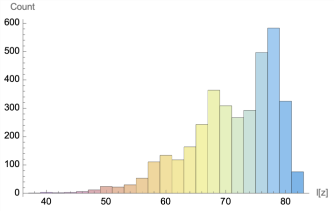
Figure 7: Side length $\frac{h}{w}$, $h = 32$, $w = 30$

Zoom in (reduce side length)

Reduce to $\frac{32}{40}$



(a) Black: $G \bigcap \hat{K}_\kappa^{(m)}$

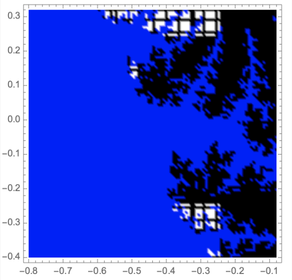


(b) Distribution of $\ell(z)$, for $z \in G \bigcap \hat{K}_\kappa$

Figure 8: Side length $\frac{h}{w}$, $h = 32$, $w = 40$

Zoom in (reduce side length)

Reduce to $\frac{32}{44}$



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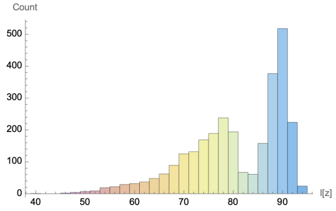
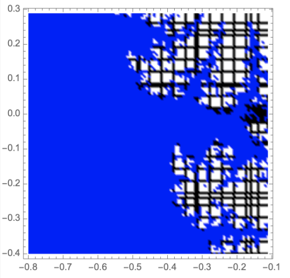


(b) Distribution of $\ell(z)$, for $z \in G \bigcap \hat{K}_\kappa$

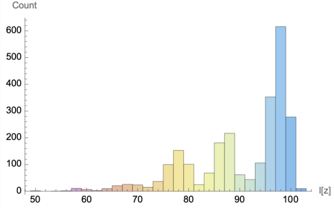
Figure 9: Side length $\frac{h}{w}$, $h = 32$, $w = 44$

Zoom in (reduce side length)

Reduce to $\frac{32}{46}$



(a) Black: $G \bigcap \hat{K}_\kappa^{(m)}$.
White: $\left\{z{:}z \in G \bigcap \hat{K}_\kappa, \ell(z) > m\right\}$

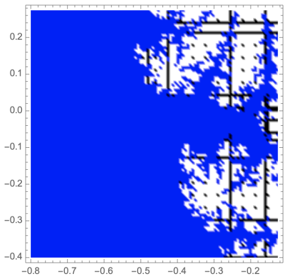


(b) Distribution of $\ell(z)$, for $z \in G \bigcap \hat{K}_\kappa$

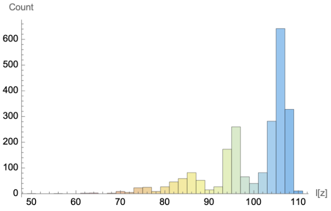
Figure 10: Side length $\frac{h}{w}$, $h = 32$, $w = 46$

Zoom in (reduce side length)

Reduce to $\frac{32}{47}$



(a) Black: $G \bigcap \hat{K}_\kappa^{(m)}$.
White: $\left\{ z{:}z \in G \bigcap \hat{K}_\kappa, \ell(z) > m \right\}$



(b) Distribution of $\ell(z)$, for $z \in G \bigcap \hat{K}_\kappa$

Figure 11: Side length $\frac{h}{w}$, $h = 32$, $w = 47$

Conclusions

Conclusions

- Inroduced a bounded complexity approximation of filled-Julia set
- Fractal oracle number Υ_{κ}
- Given the first m bits of Υ_{κ}, there is Procedure that constructs this approximation in finite time
- Υ_{κ} has complete information on how *complexity* $\ell(z)$ determines the *limiting behavior* of orbits $\mathcal{O}(z, \kappa)$, for all $z \in \hat{K}_{\kappa}$

Thanks for your attention.