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ESTIMATE OF THE NUMBER OF RESTRICTED INTEGER-PARTITIONS

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Using the saddle-point method an estimate is computed for the number $w_{\mathrm{m,N}}(n)$ of ordered m-partitions (compositions) of a positive integer n under a constraint that the size of every part is at most N. The approximation error rate is $O(n^{-1/5})$.

1. INTRODUCTION

Let n, m be positive integers. An ordered m-partition of n (also known as a *composition* [**1**]) is a sequence of positive integers $a_1, \dots, a_m$which are called *parts* such that $\sum\limits_{i=1}^{m} a_i = n$. Many applications of combinatorics involve the number of ordered m-partitions of n. This number is easy to compute. If zero-size parts are allowed then based on the generating function $1/(1-x)^m$ it is simple to show that it equals $\binom{n+m-1}{m-1}$, (see [**4**], p. 33). Note that if we are allowed to ignore the ordering of the parts then this number reduces to the classical Stirling number of the second kind (see [**3**], p. 244). Often it is the case where there is a constraint on the size of the parts. For instance, consider ordered m-partitions of n where each part must be of size between 1 and N for some constant $N \geq 1$. There are known recurrences for the number of such constrained ordered partitions (see for instance, Theorem 4.2 of [**1**]). It is not difficult to compute this exactly as the following lemma shows. First, we define the binomial coefficient quantity

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{if } 0 \leq k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

This definition is slightly more restrictive in that it does not allow the upper index n to be negative or real (for more general definitions of the binomial coefficient, see [**3**]).

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Lemma 1.1. *For $1 \leq m \leq n$, $N \geq 0$, let $w_{m,N}(n)$ be the number of ordered partitions of the integer n into m parts each of size at least 0 but no larger than N. Then*

$$w_{m,N}(n) = \sum_{i=0,N+1,2(N+1),\ldots}^{n} (-1)^{i/(N+1)} \binom{m}{i/(N+1)} \binom{n-i+m-1}{n-i}.$$

Proof. Consider the polynomial

$$(1+x+x^2+\cdots+x^N)^m = \left(\frac{1-x^{N+1}}{1-x}\right)^m. \tag{1.1}$$

The coefficient of x^nin the above polynomial gives precisely the value of $w_{n,M}(n)$. Hence we have for the generating function of $w_{m,N}(n)$

$$W(x) = \sum_{n\geq 0} w_{m,N}(n)x^n = \left(\frac{1-x^{N+1}}{1-x}\right)^m.$$

Let $T(x) = (1-x^{N+1})^m$ and $S(x) = \left(\frac{1}{1-x}\right)^m$. Then by the binomial theorem we have $T(x) = \sum_{i=0}^{m}(-1)^i \binom{m}{i} x^{i(N+1)}$ which generates the sequence $t_N(n) = \binom{m}{n/(N+1)}(-1)^{n/(N+1)}\,\mathbb{I}\,(n \bmod (N+1) = 0)$, where $\mathbb{I}(E)$ denotes the indicator function which equals 1 if the expression E is true and 0 otherwise. Similarly, for $m \geq 1$, it is easy to show $S(x)$ generates $s(n) = \binom{n+m-1}{n}$. The product $W(x) = T(x)S(x)$ generates their convolution $t_N(n) \star s(n)$, namely,

$$w(n) = \sum_{i=0,N+1,2(N+1),\ldots}^{n} (-1)^{i/(N+1)} \binom{m}{i/(N+1)} \binom{n-i+m-1}{n-i}. \tag{1.2}$$ □

Remark 1.2. The above may alternatively be expressed as

$$w_{m,N}(n) = \sum_{\ell=0}^{m}(-1)^{\ell}\binom{m}{\ell}\binom{n+m-1-\ell(N+1)}{m-1}$$

The aim of this paper is to obtain a simple and closed-form estimate of $w_{m,N}(n)$ without involving a summation operator. The following is the main result of the paper (we use the following notation to denote the Gaussian cumulative distribution, $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} \exp\{-x^2/2\}\,\mathrm{d}x$).

Theorem 1.3. *Let $n, N \geq 1$ and $m = \frac{2n}{N}\left(1+c_1 n^{-1/2}\right)$ for any absolute constant $c_1 > 0$. Denote by $\mu = N/2$ and $\sigma = (N/6)(1+N/2)$. Then as n increases, the* *number $w_{m,N}(n)$ of compositions of n into m parts each of size no larger than N satisfies*

$$w_{m,N}(n) = \frac{1}{\sqrt{2\pi m\sigma}}\,(N+1)^m \exp\left\{-\frac{(n-m\mu)^2}{2m\sigma}\right\}\left(1-2\Phi\left(-m^{1/10}\sqrt{\sigma}\right) + O\left(\exp\left(-m^{1/5}\right)\right)\right).$$

Remark 1.4. The approximation error rate is $O(n^{-1/5})$.

In the next section we present the proof. Table 1 shows a numerical example where $w_{m,N}(n)$ is approximated by $\widehat{w}_{m,N}(n)$ (the estimate of Theorem (1.3)) with $N = 6$, $c_1 = 1$, $m = \frac{2n}{N}\left(1 + c_1 n^{-1/2}\right)$ and all numerical values truncated to four significant digits.

n	m	$w_{m,N}(n)$	$\widehat{w}_{m,N}(n)$	% error
10	5	826	787	4.671
20	9	$1.385 \cdot 10^{6}$	$1.341 \cdot 10^{6}$	3.143
50	20	$1.92 \cdot 10^{15}$	$1.892 \cdot 10^{15}$	1.451
80	30	$5.423 \cdot 10^{23}$	$5.385 \cdot 10^{23}$	0.717
100	37	$4.052 \cdot 10^{29}$	$4.028 \cdot 10^{29}$	0.582
150	55	$4.885 \cdot 10^{44}$	$4.862 \cdot 10^{44}$	0.479
200	72	$1.061 \cdot 10^{59}$	$1.058 \cdot 10^{59}$	0.326

Table 1. $w_{m,N}(n)$-versus-its $w_{m,N}(n)$ versus its approximation $\widehat{w}_{m,N}(n)$, $c_1 = 1$, $N = 6$

For this example Figure 1.1 shows the numerical approximation error (in percent) versus the theoretical relative error term of $O(n^{-1/5})$.

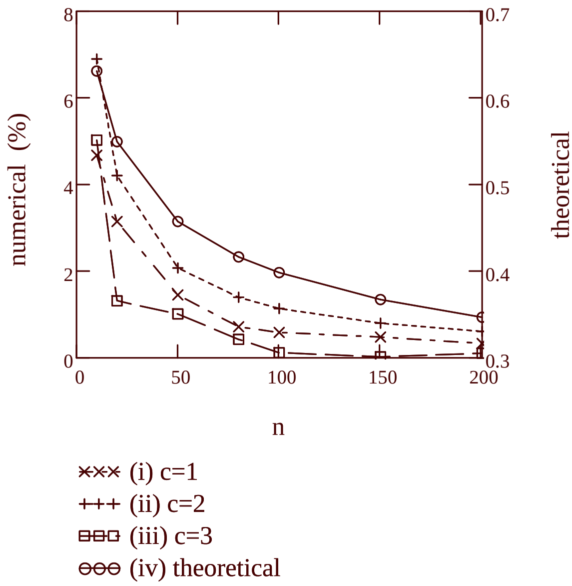

Figure 1.1. Relative error rate, numerical versus theoretical,
(i) $c_1 = 1$,
(ii) $c_1 = 2$,
(iii) $c_1 = 3$,
(iv) theoretical $O(n^{-1/5})$

2. PROOF OF THEOREM 1.3

We use the saddle point asymptotic method (see [**2**]) which is a counterpart of the LAPLACE's method for evaluation of integrals. Let $G(z)$ be analytic in a region Ω containing 0. Then by definition it has a convergent power series expansion $G(z) = \sum_{n\geq 0} c_n z^n$. By CAUCHY's coefficient formula (Theorem IV.4, [**2**]) , which is a consequence of CAUCHY's residue theorem (Theorem IV.3), the n^{th} coefficient c_n of G equals

$$c_n = \frac{1}{2\pi i} \oint_C G(z) \frac{\mathrm{d}z}{z^{n+1}}\,, \tag{2.1}$$

where C is a simple loop encircling 0 in Ω. Denote by

$$F(z) \equiv \frac{G(z)}{z^{n+1}}$$

then the method dictates to choose a contour C that passes through (or near) a saddle point ζ, i.e., a point where the derivative $F'(\zeta) = 0$ and $F(\zeta) \neq 0$. At ζ, F reaches its maximum value along C and is also the minimum of the maxima along other neighboring contours. When G has nonnegative coefficients c_n (as is the case where the coefficients express a combinatoric expression such as $w_{m,N}(n)$) there exists a saddle point on the positive real axis. It follows that a small neighborhood of the saddle point provides the dominant contribution to the integral (this is called the central part of the integral while the remaining part is called the "tail"). When this contribution can be estimated by local expansion one may resort to LAPLACE's method of integral approximation.

We now proceed with approximating this integral for our specific problem where by (1.1) we have

$$G(z) = H^m(z)$$

where

$$H(z) = \frac{1 - z^{N+1}}{1 - z}\,.$$

Clearly, we need to have

$$n \leq Nm \tag{2.2}$$

since otherwise it is not possible for m parts of size at most N to cover n. On the positive real axis F is convex since its second derivative is positive hence it has a unique minimum there. The saddle point is the solution to the equation $F'(z) = 0$ which is equivalent to solving

$$z\,\frac{G'(z)}{G(z)} = n + 1.$$

Substituting for G we obtain the following equation

$$\frac{zH'(z)}{H(z)}m = n+1. \tag{2.3}$$

Take

$$m = \frac{2n}{N}\left(1 + c_1 n^{-1/2}\right), \tag{2.4}$$

for $c_1 > 0$ any absolute constant. We have $H'(1)/H(1) = N/2$ hence, when $z = 1$, the left side of (2.3) equals $n\left(1 + c_1 n^{-1/2}\right)$ which for large n is close to n. Hence we take $z = 1$ as an approximation to the saddle point (solution of (2.3)) and choose the contour C to be a circle of radius 1 centered at the origin.

In polar coordinates we have $z = re^{i\theta}$and $dz = rie^{i\theta}$ thus substituting for $r = 1$, the integral (2.1) becomes

$$\begin{aligned}\frac{1}{2\pi i}\oint_C \left(\frac{1-z^{N+1}}{1-z}\right)^m \frac{\mathrm{d}z}{z^{n+1}} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1-e^{i(N+1)\theta}}{1-e^{i\theta}}\right)^m e^{-in\theta}\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{m\ln g(\theta)}e^{-in\theta}\mathrm{d}\theta,\end{aligned}$$

where

$$g(\theta) = \frac{1-e^{i(N+1)\theta}}{1-e^{i\theta}}. \tag{2.5}$$

We split the integral into two parts as follows:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{m\ln g(\theta)}e^{-in\theta}\mathrm{d}\theta = \underbrace{\frac{1}{2\pi}\int_{-\theta_0}^{\theta_0} e^{m\ln g(\theta)}e^{-in\theta}\mathrm{d}\theta}_{\text{(I)}} + \underbrace{\frac{1}{2\pi}\int_{\theta_0}^{2\pi-\theta_0} e^{m\ln g(\theta)}e^{-in\theta}\mathrm{d}\theta}_{\text{(II)}} \tag{2.6}$$

where we later choose θ_0 to go to zero such that the first (central) part is dominant and the second (tail) is negligible compared to the total with increasing n, the exact rate will later be shown to satisfy

$$m\theta_0^2 \to +\infty,\; m\theta_0^3 \to 0. \tag{2.7}$$

We start with analyzing integral (II). Represent the complex number g by $g(\theta) = R(\theta)\exp\left(iT(\theta)\right)$ with

$$\ln(g) = \ln R + iT \tag{2.8}$$

then we have

$$R^2(\theta) = \frac{1-\cos\big((N+1)\theta\big)}{1-r\cos(\theta)}. \tag{2.9}$$

As we now show, the tail integral (II) is negligible compared to the total integral (I) + (II). Denote by the ratio

$$\rho \equiv \left| \frac{\int_{\theta_0}^{2\pi-\theta_0} \exp\big\{m\ln g(\theta) - in\theta\big\}\,\mathrm{d}\theta}{\int_{-\theta_0}^{\theta_0} \exp\big\{m\ln g(\theta) - in\theta\big\}\,\mathrm{d}\theta + \int_{\theta_0}^{2\pi-\theta_0} \exp\big\{m\ln g(\theta) - in\theta\big\}\,\mathrm{d}\theta} \right| \tag{2.10}$$

then it suffices to show that $\rho \ll 1$. Since θ_0 is small, then over the interval $[0,\theta_0]$, R strictly decreases and for all remaining θ its value never surpasses $R(\theta_0)$ hence $|g(\theta)| \le R(\theta_0)$ for $\theta \in [\theta_0, 2\pi-\theta_0]$ and the numerator of (2.10) is bounded from above by $2\pi\exp\big\{m\ln R(\theta_0)\big\}$. Dividing both numerator and denominator of (2.10) we obtain

$$\rho \le 2\pi \Big/ \left| \int_{-\theta_0}^{\theta_0} \exp\left\{m\ln\left(\frac{g(\theta)}{R(\theta_0)}\right)\right\}\exp\{-in\theta\}\,\mathrm{d}\theta \right.$$
$$\left. + \int_{\theta_0}^{2\pi-\theta_0} \exp\left\{m\ln\left(\frac{g(\theta)}{R(\theta_0)}\right)\right\}\exp\{-in\theta\}\,\mathrm{d}\theta \right|.$$

Bounding from above the absolute value of the second integral and, for the first integral, using $\Re\{\exp\{-in\theta\}\} \ge \Re\{\exp\{-in\theta_0\}\}$, $\Im\{\exp\{-in\theta\}\} \ge \Im\{\exp\{-in\theta_0\}\}$ for $\theta \in [-\theta_0, \theta_0]$ (where $\Re$ and $\Im$ denote the real and imaginary parts) we obtain

$$\rho \le \frac{2\pi}{\left|\exp\{-in\theta_0\}\int_{-\theta_0}^{\theta_0}\exp\left\{m\ln\left(\frac{g(\theta)}{R(\theta_0)}\right)\right\}\mathrm{d}\theta\right| - 2(\pi-\theta_0)}.$$

For the first term in the denominator above we have

$$\left|\exp\{-in\theta_0\}\int_{-\theta_0}^{\theta_0}\exp\left\{m\ln\left(\frac{g(\theta)}{R(\theta_0)}\right)\right\}\mathrm{d}\theta\right| = \left|\int_{-\theta_0}^{\theta_0}\exp\left\{m\ln\left(\frac{g(\theta)}{R(\theta_0)}\right)\right\}\mathrm{d}\theta\right|$$

which equals

$$\left|\int_{-\theta_0}^{\theta_0}\exp\{m\ln(R_0(\theta))\}\exp\{imT(\theta)\}\,\mathrm{d}\theta\right|, \tag{2.11}$$

where we denote by $R_0(\theta) \equiv R(\theta)/R(\theta_0)$. From (2.5) we may express $g(\theta)$ as

$$\left(\frac{\exp\left\{i\,\frac{(N+1)\theta}{2}\right\}}{\exp\left\{i\,\frac{\theta}{2}\right\}}\right)\frac{\exp\left\{i\,\frac{(N+1)\theta}{2}\right\}-\exp\left\{-i\,\frac{(N+1)\theta}{2}\right\}}{\exp\left\{i\,\frac{\theta}{2}\right\}-\exp\left\{-i\,\frac{\theta}{2}\right\}}$$
$$=\frac{\exp\left\{i\,\frac{(N+1)\theta}{2}\right\}}{\exp\left\{i\,\frac{\theta}{2}\right\}}\,\frac{\sin\left(\frac{(N+1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}.$$

Clearly, the polar angle of this expression is

$$T(\theta) = \frac{N\theta}{2} \tag{2.12}$$

which is an odd function of θ. Together with $R_0(\theta)$ being an even function it follows that the right side of (2.11) equals

$$2\left|\int_0^{\theta_0} \exp\{m\ln(R_0(\theta))\}\cos(mN\theta/2)\,\mathrm{d}\theta\right|. \tag{2.13}$$

Next, from a Maclaurin series expansion of $R_0(\theta)$ we determine that $R_0(\theta)$ is concave on $[0,\theta_0]$ hence is bounded there from below by a linear function

$$\ell(\theta) = \frac{1-R_0(0)}{\theta_0}\,\theta + R_0(0).$$

In (2.13) the integrand is positive hence the expression is bounded from below by

$$2\exp\big\{m\ln\big(R_0(0)\big)\big\}\left|\int_0^{\theta_0}\exp\{m\ln(1-b\theta)\}\cos(mN\theta/2)\,\mathrm{d}\theta\right| \tag{2.14}$$

where $b \equiv \big(R_0(0)-1\big)/\big(\theta_0 R_0(0)\big)$ which is positive since $R_0(0) \geq 1$. Expanding the log factor in θ we obtain for the integral above

$$\int_0^{\theta_0}\exp\{-m\gamma b\theta\}\cos(mN\theta/2)\,\mathrm{d}\theta$$

for some absolute constant $\gamma > 0$. Letting $u = m\theta$ and changing variables this becomes

$$\frac{1}{m}\int_0^{m\theta_0}\exp\{-\gamma b u\}\cos(Nu/2)\,\mathrm{d}u$$
$$=\left(\frac{(2N\sin(Nm\theta_0/2)-4\gamma b\cos(Nm\theta_0/2))\exp\{-\gamma b m\theta_0\}+4\gamma b}{4\gamma^2b^2+N^2}\right)\frac{1}{m}.$$

The first factor is bounded from below by a constant $c_2 > 0$ independent of m. Hence the expression in (2.14) is bounded from below by

$$2c_2 m^{-1} \exp\left\{m \ln\left(R_0(0)\right)\right\}.$$

It follows from the above that

$$\rho = O\big(m \exp\{-c_3 m\}\big)$$

for some constant $c_3 > 0$ independent of m. Hence we showed that the tail integral (I) is negligible compared to the total sum of (I) and (II). We continue now to analyze integral (I). Using (2.8) we have

$$\frac{1}{2\pi}\int_{-\theta_0}^{\theta_0} e^{m\ln g(\theta)} e^{-in\theta} \mathrm{d}\theta = \frac{1}{2\pi}\int_{-\theta_0}^{\theta_0} e^{\frac{m}{2}\ln R^2} e^{-i(n\theta - mT)} \mathrm{d}\theta. \tag{2.15}$$

We may expand $\ln R^2(\theta)$ into series around the origin to obtain

$$\ln R^2(\theta) = 2\ln(N+1) - \frac{N}{6}\left(\frac{N}{2}+1\right)\theta^2 + O(\theta^3).$$

Letting

$$\sigma \equiv \sigma(r,N) = \frac{N}{6}\left(\frac{N}{2}+1\right) \tag{2.16}$$

then (2.15) equals

$$\frac{1}{2\pi}(N+1)^m \left(1 + O(m\theta_0^{\,3})\right)\left(\int_{-\theta_0}^{\theta_0} \exp\left\{-\frac{m\sigma\theta^2}{2}\right\}\cos\left(n\theta - mT(\theta)\right)\mathrm{d}\theta \right. \\ \left. - i\int_{-\theta_0}^{\theta_0} \exp\left\{-\frac{m\sigma\theta^2}{2}\right\}\sin\left(n\theta - mT(\theta)\right)\mathrm{d}\theta\right), \tag{2.17}$$

where we used $\exp\left\{-mO(\theta^3)/2\right\} = 1 + O(m\theta^3)$ for $|\theta| \le \theta_0$. As was shown above, $T(\theta) = N\theta/2$ is odd hence it follows that the second integral in (2.17) vanishes. Hence we are left with approximating the first integral

$$\int_{-\theta_0}^{\theta_0} \exp\left\{-\frac{m\sigma\theta^2}{2}\right\}\cos\left(\theta(n - mN/2)\right)\mathrm{d}\theta. \tag{2.18}$$

A second order approximation of the cosine is not appropriate here since its frequency may increase faster than the rate of decrease of θ_0. This would yield an approximation (based on a negative quadratic) which is too loose over the domain of integration. Instead, we state an auxiliary lemma that allows us to obtain a more accurate approximation of the integral in (2.18). Let $\Phi(x)$ denote the Gaussian cumulative probability distribution function.

Lemma 2.1. *Let $a > 0$ be fixed. Then the integral*

$$I = \int_{-L}^{L} \exp\left\{-ax^2\right\} \cos x \,\mathrm{d}x$$

equals

$$\sqrt{\frac{\pi}{a}} \exp\left\{-\frac{1}{4a}\right\} \left(1 - 2\Phi\left(-\sqrt{2a}L\right) + O\left(\exp\left\{-L^2 a\right\}\right)\right)$$

with $L \to +\infty$.

Proof. First, we have

$$\begin{aligned} \int_{-L}^{L} \exp\left\{-ax^2\right\} \cos x \,\mathrm{d}x &= \frac{1}{2} \exp\left\{-\frac{1}{4a}\right\} \left(\int_{-L}^{L} \exp\left\{-\left(\sqrt{a}x - \frac{i}{2\sqrt{a}}\right)^2\right\} \mathrm{d}x \right. \\ &\qquad \left. + \int_{-L}^{L} \exp\left\{-\left(\sqrt{a}x + \frac{i}{2\sqrt{a}}\right)^2\right\} \mathrm{d}x \right) \\ &= \frac{1}{2\sqrt{a}} \exp\left\{-\frac{1}{4a}\right\} \left(\int_{-\sqrt{a}L}^{\sqrt{a}L} \exp\left\{-\left(x - \frac{i}{2\sqrt{a}}\right)^2\right\} \mathrm{d}x \right. \\ &\qquad \left. + \int_{-\sqrt{a}L}^{\sqrt{a}L} \exp\left\{-\left(x + \frac{i}{2\sqrt{a}}\right)^2\right\} \mathrm{d}x \right). \end{aligned} \tag{2.19}$$

Consider the rectangle with vertices $(-L\sqrt{a}, 0)$, $(L\sqrt{a}, 0)$, $(L\sqrt{a}, -1/(2\sqrt{a})$, and $(-L\sqrt{a}, -1/(2\sqrt{a}))$. We integrate the function $\exp\left\{-az^2\right\}$ over the complex plane where the rectangle serves as the contour of integration:

$$\begin{aligned} &\int_{-L\sqrt{a}}^{L\sqrt{a}} \exp\left\{-x^2\right\} \mathrm{d}x + \int_{L\sqrt{a}}^{-L\sqrt{a}} \exp\left\{-\left(x - \frac{i}{2\sqrt{a}}\right)^2\right\} \mathrm{d}x \\ &+ \int_{0}^{-1/(2\sqrt{a})} \exp\left\{-\left(L\sqrt{a} + iy\right)^2\right\} \mathrm{d}y + \int_{-1/(2\sqrt{a})}^{0} \exp\left\{-\left(-L\sqrt{a} + iy\right)^2\right\} \mathrm{d}y. \end{aligned} \tag{2.20}$$

Since $\exp\left\{-az^2\right\}$ is analytic inside and on the rectangle it follows by the Null Integral property (Theorem IV.2, [**2**]) that the sum above equals zero. We now show that each of the last two integrals converges to zero with increasing L. The first of the two can be bounded as follows,

$$\text{(2.21)}\quad \int_0^{-1/(2\sqrt{a})} \exp\left\{-\left(L\sqrt{a}+iy\right)^2\right\} \mathrm{d}y \le \int_0^{-1/(2\sqrt{a})} \left|\exp\left\{-\left(L\sqrt{a}+iy\right)^2\right\}\right| \mathrm{d}y$$

$$= \int_0^{-1/(2\sqrt{a})} \left|\exp\left\{-L^2a+y^2-i2yL\sqrt{a}\right\}\right| \mathrm{d}y$$

$$= \exp\left\{-L^2a\right\} \int_0^{-1/(2\sqrt{a})} \exp\left\{y^2\right\} \mathrm{d}y$$

which is $O\left(\exp\left\{-L^2a\right\}\right)$ since the integral in (2.21) is finite (using the given condition that $a > 0$ is fixed). Similarly, the second of the two integrals can be shown to vanish at the same rate. Together with (2.20) and Cauchy's residue theorem it follows that

$$\text{(2.22)}\quad \int_{-L\sqrt{a}}^{L\sqrt{a}} \exp\left\{-x^2\right\} \mathrm{d}x = \int_{-L\sqrt{a}}^{L\sqrt{a}} \exp\left\{-\left(x-\frac{i}{2\sqrt{a}}\right)^2\right\} \mathrm{d}x + O\left(\exp\left\{-L^2a\right\}\right).$$

Now, consider a similar rectangle with vertices

$$(-L\sqrt{a},0),\ (L\sqrt{a},0),\ (L\sqrt{a},1/(2\sqrt{a}\,)),\ (-L\sqrt{a},1/(2\sqrt{a}\,)).$$

We integrate the function $\exp\left\{-az^2\right\}$ over the complex plane where the rectangle serves as the contour of integration:

$$\text{(2.23)}\quad \int_{-L\sqrt{a}}^{L\sqrt{a}} \exp\left\{-x^2\right\} \mathrm{d}x + \int_{L\sqrt{a}}^{-L\sqrt{a}} \exp\left\{-\left(x+\frac{i}{2\sqrt{a}}\right)^2\right\} \mathrm{d}x$$

$$+ \int_0^{1/(2\sqrt{a})} \exp\left\{-\left(L\sqrt{a}+iy\right)^2\right\} \mathrm{d}y + \int_{1/(2\sqrt{a})}^{0} \exp\left\{-\left(-L\sqrt{a}+iy\right)^2\right\} \mathrm{d}y$$

which equals zero by the same reasoning as above. The last two integrals above also vanish at the rate of $O\left(\exp\left\{-L^2a\right\}\right)$ so it follows that

$$\text{(2.24)}\quad \int_{-L\sqrt{a}}^{L\sqrt{a}} \exp\left\{-x^2\right\} \mathrm{d}x = \int_{-L\sqrt{a}}^{L\sqrt{a}} \exp\left\{-\left(x+\frac{i}{2\sqrt{a}}\right)^2\right\} \mathrm{d}x + O\left(\exp\left\{-L^2a\right\}\right).$$

From (2.19), (2.22) and (2.24) it follows that

$$(2.25)\quad \int_{-L}^{L} \exp\left\{-ax^2\right\}\cos x\,\mathrm{d}x = \frac{1}{\sqrt{a}}\exp\left\{-\frac{1}{4a}\right\}\left(\int_{-L\sqrt{a}}^{L\sqrt{a}} \exp\left\{-x^2\right\}\mathrm{d}x + O\left(\exp\left\{-L^2a\right\}\right)\right).$$

The integral on the right may be expressed as

$$(2.26)\quad \int_{-L\sqrt{a}}^{L\sqrt{a}} \exp\left\{-x^2\right\}\mathrm{d}x = \frac{1}{\sqrt{2}}\int_{-L\sqrt{2a}}^{L\sqrt{2a}} \exp\left\{-\frac{x^2}{2}\right\}\mathrm{d}x.$$

By definition of the Gaussian distribution function Φ it follows that the integral on the right equals

$$(2.27)\quad \sqrt{\pi}\left(1-2\Phi\left(-\sqrt{2a}L\right)\right).$$

From (2.25), (2.26) and (2.27) the statement of the lemma follows. □

We continue with the proof of the theorem. Denote by $\Delta = |n - mN/2|$ and $a = (m\sigma)/(2\Delta^2)$ then the expression in (2.18) can be written as

$$(2.28)\quad \int_{-\theta_0}^{\theta_0} \exp\left\{-\frac{m\sigma\theta^2}{2}\right\}\cos\big(\theta(n-mN/2)\big)\,\mathrm{d}\theta = \int_{-\theta_0}^{\theta_0} \exp\left\{-\frac{m\sigma\theta^2}{2}\right\}\cos(\theta\Delta)\,\mathrm{d}\theta = \frac{1}{\Delta}\int_{-\theta_0\Delta}^{\theta_0\Delta} \exp\left\{-a\alpha^2\right\}\cos\alpha\,\mathrm{d}\alpha.$$

Let us choose $\theta_0 = m^{-2/5}$ hence $m\theta_0^2 \to +\infty$, $m\theta_0^3 \to 0$ and therefore the $O(m\theta_0^3)$ term in (2.17) vanishes with increasing m. Hence we have

$$(2.29)\quad \sqrt{a}\theta_0\Delta = \sqrt{\frac{m\sigma}{2}}\,m^{-2/5} = m^{1/10}\sqrt{\frac{\sigma}{2}}$$

which increases to infinity with m and n (by (2.4)). A crucial fact is that a does not decrease arbitrarily close to zero with increasing n and rather can be bounded from below by some fixed constant. To see this, using (2.4), plug the values for m and Δ to get

$$a = \frac{m\sigma}{2\Delta^2} = \frac{m\sigma}{2\left|n - mN/2\right|^2} = \frac{n(1+c_1n^{-1/2})\sigma}{N\left|n-(1+c_1n^{-1/2})n\right|^2} = \frac{\sigma(1+c_1n^{-1/2})}{Nc_1^2}$$

where the right side is strictly greater than zero for all n. Thus we may resort to Lemma 2.1, letting $L = \theta_0 \Delta$, and obtain that the expression in (2.28) is asymptotically equal to

$$\sqrt{\frac{\pi}{a}} \, \frac{\exp\left\{-\frac{1}{4a}\right\}}{\Delta} \left(1 - 2\Phi\left(-\sqrt{2a}\theta_0\Delta\right) + O\left(\exp\left\{-\left(\theta_0\Delta\right)^2 a\right\}\right)\right).$$

Upon substituting for a and Δ in this expression, then together with (2.17) and (2.18) we obtain as an estimate of $w_{m,N}(n)$ the following expression,

$$\frac{1}{\sqrt{2\pi m\sigma}} \left(N+1\right)^m \exp\left\{-\frac{(n - mN/2)^2}{2m\sigma}\right\} \left(1 - 2\Phi\left(-m^{1/10}\sqrt{\sigma}\right) + O\left(\exp\left\{-m^{1/5}\right\}\right)\right), \tag{3.30}$$

with σ as defined in (2.16). It follows that the sum of (I) and (II) in (2.6) and hence the integral (2.1) takes this value asymptotically with n.

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