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On the Approximation of Functional Classes Equipped with a Uniform Measure Using Ridge Functions*

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We introduce a construction of a uniform measure over a functional class $\mathscr{B}^r$ which is similar to a Besov class with smoothness index r. We then consider the problem of approximating $\mathscr{B}^r$ using a manifold M_n which consists of all linear manifolds spanned by n ridge functions, i.e., $M_n = \{\sum_{i=1}^n g_i(a_i \cdot x) \colon a_i \in S^{d-1}, g_i \in L_2([-1, 1])\}$, $x \in B^d$. It is proved that for some subset $A \subset \mathscr{B}^r$ of probabilistic measure $1-\delta$, for *all* $f \in A$ the degree of approximation of M_n behaves asymptotically as $1/n^{r/(d-1)}$. As a direct consequence the probabilistic (n, δ)-width for *nonlinear* approximation denoted as $d_{n,\delta}(\mathscr{B}^r, \mu, M_n)$, where μ is a uniform measure over $\mathscr{B}^r$, is similarly bounded. The lower bound holds also for the specific case of approximation using a manifold of one hidden layer neural networks with n hidden units. © 1999 Academic Press

1. INTRODUCTION

We consider the problem of approximating a functional class $\mathscr{B}^r$ similar to a Besov class using a manifold of ridge functions $M_n = \{\sum_{i=1}^n g_i(a_i \cdot x) \colon a_i \in S^{d-1}, g_i \in L_2([-1, 1])\}$, defined on the unit ball $B^d = \{x \in \mathbb{R}^d \colon \|x\|_2 := (\sum_{i=1}^d x_i^2)^{1/2} \leqslant 1\}$ in the space $\mathbb{R}^d$. Here $S^{d-1} = \{x \in B^d \colon \|x\|_2 = 1\}$ is the

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unit sphere in $\mathbb{R}^d$. The degree of approximation of $f \in \mathcal{B}^r$ by M_n in the space L_2 is defined by the expression

$$\operatorname{dist}(f, M_n, L_2) = \inf_{h \in M_n} \|f - h\|_{L_2},$$

where $\|f\|_{L_2}$ denotes the L_2-norm of f on B^d.

Vostrecov and Kreines [27] and Lin and Pinkus [8, 9] studied issues of fundamentality of ridge functions in functional spaces. The specific case in which the ridge functions are sigmoidal, e.g., $g_i(y) = \sigma(y) = 1/1 + e^{-y}$, $1 \leqslant i \leqslant n$, and translations are permitted corresponds to a manifold $\mathcal{H}_n = \{\sum_{i=1}^n c_i \sigma(a_i \cdot x + b_i) : a_i \in \mathbb{R}^d, b_i, c_i \in \mathbb{R}\}$, of one hidden layer neural networks with n hidden units. There have been many investigations concerning the approximation properties of $\mathcal{H}_n$, e.g., Barron [1], Mhaskar [15], Girosi *et al.* [4–6], DeVore *et al.* [2], Petrushev [19], and Maiorov and Meir [13].

Recently a series of results was obtained for estimates of approximation of functions by the ridge-manifold M_n in the two-dimensional case, $d = 2$ (see Oskolkov [18], and Temlyakov [23]). In particular, Oskolkov showed that for $d = 2$ the orders of approximation of radial functions by the ridge-manifold M_n and by the space of algebraic polynomials of degree n coincide. In Maiorov [12], the asymptotic behavior of the distance

$$\operatorname{dist}(W_2^{r,d}, M_n, L_2) \asymp n^{-r/(d-1)}$$

for the Sobolev class $W_2^{r,d}$, $d \geqslant 2$, was obtained.

In this work we are interested in assessing how massive is the subset of functions in $\mathcal{B}^r$ such that for *all* functions in this subset a certain degree of approximation holds. In order to formalize the statement that a high percentage of the functions in $\mathcal{B}^r$ are approximated by M_n to a certain degree we construct a uniform measure over $\mathcal{B}^r$. The volume of a subset A of the unit ball in $\mathbb{R}^n$ which is equipped with a uniform probability measure is proportional to the probability of A. Similarly if $\mathcal{B}^r$ is equipped with the uniform measure then a subset in $\mathcal{B}^r$ of high probability is interpreted as being massive in the sense of occupying almost all of $\mathcal{B}^r$.

To proceed we first construct a uniform measure μ over $\mathcal{B}^r$. We then calculate lower and upper bounds on the degree of approximation by M_n which holds for all functions in some subset $A \subset \mathcal{B}^r$ of probability $1 - \delta$. Specifically we obtain a degree of approximation such that for some $A \subset \mathcal{B}^r$, with $\mu(A) \geqslant 1 - e^{-\alpha(n)}$ and $\alpha(n) = c_0 n^{d/(d-1)}$, then for all $f \in A$, $c_1/n^{r/(d-1)} \leqslant \operatorname{dist}(f, M_n, L_2) \leqslant c_2/n^{r/(d-1)}$, for some constants c_0, $c_1, c_2 > 0$ depending on r and d, but not on n.

In [12], upper and lower bounds on the distance

$$\operatorname{dist}(W_2^{r,d}, M_n, L_2) = \sup_{f \in W_2^{r,d}} \operatorname{dist}(f, M_n, L_2)$$

for a Sobolev class $W_2^{r,d}$ were obtained. However, this type of result only guarantees the existence of a function in $W_2^{r,d}$ for which the lower bound holds. That is, it is a "worst case" result. In this work we extend that result by obtaining tight lower and upper bounds that hold for all functions in a subset of large measure in $\mathscr{B}^r$.

As a consequence, we obtain asymptotically tight lower and upper bounds on the distance between $\mathscr{B}^r$ and M_n measured by a probabilistic (n, δ)-width which is defined as

$$d_{n,\delta}(\mathscr{B}^r, \mu, M_n) = \inf_{\substack{A \subset \mathscr{B}^r \\ \mu(A) = 1-\delta}} \sup_{f \in A} \operatorname{dist}(f, M_n, L_2), \tag{1}$$

where $0 \leqslant \delta \leqslant 1$ and the infimum runs over all subsets A of $\mathscr{B}^r$ with probability $\mu(A) = 1 - \delta$. From the construction of the class $\mathscr{B}^r$ one can see that for any $0 \leqslant \delta \leqslant 1$ there exists a subset $A \in \mathscr{B}^r$ such that $\mu(A) = 1 - \delta$. Quantities similar to (1) were considered in [25, 11, 14] where μ was taken to be a Gaussian or Wiener measure and the approximation was linear.

From (1) the next inverse formulation follows

$$\mu\{f \in \mathscr{B}^r : \operatorname{dist}(f, M_n, L_2) \geqslant d_{n,\delta}\} = \delta,$$

where $d_{n,\delta} = d_{n,\delta}(\mathscr{B}^r, \mu, M_n)$. Indeed, from (1) it follows that there exists the subset A in $\mathscr{B}^r$ such that $\mu(A) = 1 - \delta$ and

$$\begin{aligned} &\mu\{f \in \mathscr{B}^r : \operatorname{dist}(f, M_n, L_2) \geqslant d_{n,\delta}\} \\ &\quad = \mu\{f \in \mathscr{B}^r : \operatorname{dist}(f, M_n, L_2) \geqslant \sup_{h \in A} \operatorname{dist}(h, M_n, L_2)\} \end{aligned}$$

and hence

$$\mu\{f \in \mathscr{B}^r : \operatorname{dist}(f, M_n, L_2) \geqslant d_{n,\delta}\} = \mu\{\mathscr{B}^r \backslash A\} = 1 - \mu(A) = \delta.$$

The main contributions of this paper are threefold: (i) the construction of a uniform measure over a functional class $\mathscr{B}^r$ which is similar to a Besov class. (ii) Proving a lower bound on the degree of approximation by ridge functions which holds for all functions in some subset of $\mathscr{B}^r$ of probability measure $1 - \delta$ with respect to the uniform measure. (iii) Introducing a probabilistic width $d_{n,\delta}$ for *nonlinear* approximation and estimating $d_{n,\delta}(\mathscr{B}^r, \mu, M_n)$ for a uniform measure μ.

2. PRELIMINARIES

We begin by introducing some notation. For an integer $m \geqslant 1$ let $\mathbb{Z}_m = \{1, 2, ..., m\}$. Consider the ball of radius r in $\mathbb{R}^m$ denoted by $B^m(r) = \{x \in \mathbb{R}^m : \|x\|_2 \leqslant r\}$ and set $B^m = B^m(1)$. For a vector $z \in \mathbb{R}^m$, $\text{sgn}(z) = (\text{sgn}(z_1), ..., \text{sgn}(z_m))$, $\text{sgn}(z_i) = 1$ for $z_i \geqslant 0$, $\text{sgn}(z_i) = -1$ for $z_i < 0$. We denote by $\|v\|_{l_p^m}$ or simply $\|v\|_p$, $p \geqslant 1$, the l_p^m Euclidean norm of $v \in \mathbb{R}^m$. For any Euclidean sets A and B in $\mathbb{R}^m$ we use a distance function $\text{dist}(a, B, l_p^m) = \inf_{b \in B} \|a - b\|_{l_p^m}$ for any $a \in A$, and $\text{dist}(A, B, l_p^m) = \sup_{a \in A} \text{dist}(a, B, l_p^m)$.

Define the space of functions

$$L_2 = L_2(B^d) = \left\{ f : \|f\|_{L_2} := \left(\int_{B^d} |f(x)|^2 \, dx \right)^{1/2} < \infty \right\}.$$

We write $\int_{B^d} f(x)\, dx$ where $x = (x_1, ..., x_d)$, and $dx = dx_1 \cdots dx_d$.

The notation $a_n \asymp b_n$ in this paper means that there exist constants $c_1, c_2 > 0$ which depend only on the smoothness parameter r of the class $\mathscr{B}^r$ and the dimensionality d of the domain B^d such that for every $n \geqslant 1$, $c_1 \leqslant a_n / b_n \leqslant c_2$.

We define the class of functions $\mathscr{B}^r$ using the classical means of approximation, namely, algebraic polynomials. Consider the space $\mathscr{P}_s = \text{span}\{x_1^{k_1} \cdots x_d^{k_d} : |k| = k_1 + \cdots + k_d \leqslant s\}$, $s = 0, 1, ...,$ consisting of all algebraic polynomials on $\mathbb{R}^d$ of total degree at most s. Let $\mathscr{P}_s^h = \text{span}\{x_1^{k_1} \cdots x_d^{k_d} : |k| = s\}$ be the subspace of $\mathscr{P}_s$ consisting of homogeneous polynomials of degree s. Set $m_s = \dim \mathscr{P}_s^h$. It is known (cf. [22]) that $m_s = \binom{d+s-1}{d-1} \asymp s^{d-1}$.

Let the set of polynomials $Q_s = \{q_l\}_{l=1}^{m_s}$ be a basis in $\mathscr{P}_s^h$. The set of polynomials $\bigcup_{s=0}^{\infty} Q_s$ is a complete system of functions in the space L_2. Using the method of orthogonalization in L_2 we can construct a complete orthogonal system of polynomials in L_2

$$P = \bigcup_{s=0}^{\infty} \{p_{s,1}, ..., p_{s,m_s}\},$$

such that the set $P_s^h = \{p_{s,1}, ..., p_{s,m_s}\}$ is a complete orthonormal system of functions in the subspace $\mathscr{P}_s^h$. Note in particular that in [12] we constructed one specific orthonormal system of algebraic polynomials in L_2.

For any natural N we denote the set of multi-indexes

$$\Delta_N = \{(s, l) : s = 2^N + 1, ..., 2^{N+1}, l = 1, ..., m_s\}.$$

Introduce the subspace $\Phi_N = \operatorname{span}\{p_{s,l} : (s, l) \in \Delta_N\}$. Let G_N^r, $r > 0$, be the ball with radius 2^{-rN} in the space Φ_N, that is,

$$G_N^r = \left\{ \sum_{(s,l) \in \Delta_N} c_{s,l} p_{s,l} \in \Phi_N : \left(\sum_{(s,l) \in \Delta_N} |c_{s,l}|^2 \right)^{1/2} \leqslant 2^{-rN} \right\}.$$

Denote by $\mathscr{B}^r$, the set of all functions $f \in L_2(B^d)$ which can represented as infinite sums of functions from G_N^r, namely

$$\mathscr{B}^r = \left\{ f : f = \sum_{N=0}^{\infty} f_N, f_N \in G_N^r, N = 0, 1, ... \right\}.$$

It is not hard to see that the class $\mathscr{B}^r$ is essentially equivalent to the class H^r, consisting of all functions f for which the best approximation by algebraic polynomials of degree 2^N satisfies the inequality

$$\operatorname{dist}(f, \mathscr{P}_{2^N}, L_2) \leqslant 2^{-rN} \qquad (N = 0, 1, ...).$$

From Jackson's Theorem (see [24]), it follows that the Sobolev class $W_2^{r,d}$ belongs to the class H^r and hence also to the class $c\mathscr{B}^r$, for some constant c. Observe also that the latter class (discussed also in [23]) is analogous to the Besov class [26] which is defined using trigonometric polynomials.

As an approximating function class we will use the following nonlinear manifold

$$M_n = \left\{ h(x) = \sum_{l=1}^{n} h_l(a_l \cdot x) : a_l \in S^{d-1}, h_l \in L_2([-1, 1]) \right\} \qquad (x \in B^d) \qquad (2)$$

which represents the union of all linear manifolds that are spanned by n ridge functions from the space $L_2([-1, 1])$ of square-integrable functions on the segment $[-1, 1]$.

3. UNIFORM MEASURE CONSTRUCTION

The construction of a uniform measure over a functional class is nontrivial. For example, it is not possible to construct such a measure over a Sobolev or Besov class. For this reason we consider the class $\mathscr{B}^r$ which permits such a construction.

Let $P = \{p_{s,l}\}$ be a complete system of orthonormal polynomials in L_2, as constructed in Section 2. Then we can express the class $\mathscr{B}^r$ as

$$\mathscr{B}^r = \Bigg\{ f \in L_2 \colon f(x) = \sum_{N=0}^{\infty} \sum_{(s,l) \in \Delta_N} c_{s,l} p_{s,l}(x), \\ \left(\sum_{(s,l) \in \Delta_N} |c_{s,l}|^2 \right)^{1/2} \leqslant 2^{-rN}, \text{ for all } N \geqslant 0 \Bigg\}. \tag{3}$$

Consider the subspace $\Phi_N = \operatorname{span}\{p_{s,l} \colon (s,l) \in \Delta_N\}$. We have that Φ_N is orthogonal to $\Phi_{N'}$, for all $N \neq N'$, and $\mathscr{B}^r$ is isomorphic to the set D^r of inifinite sequences of finite dimensional vectors, i.e.,

$$\begin{aligned} \mathscr{B}^r \simeq D^r &:= \prod_{N=0}^{\infty} B^{|\Delta_N|}(2^{-rN}) \\ &:= \{c = (c^0, ..., c^N, ...) \colon c^N \in B^{|\Delta_N|}(2^{-rN})\}, \end{aligned} \tag{4}$$

where $c^N := (c_{s,l})_{(s,l) \in \Delta_N}$, and $|\Delta_N|$ is the cardinality of Δ_N, $N \geqslant 0$.

Note that the cardinality of Δ_N satisfies the asymptotic

$$|\Delta_N| = \sum_{s=2^N+1}^{2^{N+1}} \dim \mathscr{P}_s^h = \sum_{s=2^N+1}^{2^{N+1}} m_s \asymp \sum_{s=2^N+1}^{2^{N+1}} s^{d-1} \asymp 2^{dN}.$$

Let $b_n \equiv B^{|\Delta_n|}(2^{-rn})$ be the ball of radius 2^{-rn} in $\mathbb{R}^{|\Delta_n|}$, and denote the volume of b_n by $\operatorname{vol}(b_n)$. Let $v_n(dc^n) = dc^n/\operatorname{vol}(b_n)$ be the normed Lebesgue measure on b_n, $v_n(b_n) = 1$, and

$$D_N^r = \prod_{n=0}^{N} b_n.$$

For $c = (c^0, ..., c^N) \in D_N^r$ define the measure on D_N^r as

$$\lambda_N(dc) = \prod_{n=0}^{N} v_n(dc^n).$$

Now, let $B \subset D_N^r$. We have

$$\begin{aligned} \lambda_{N+1}(B \times b_{N+1}) &= \int_{B \times b_{N+1}} \lambda_N(dc)\, v_{N+1}(dc^{N+1}) \\ &= \frac{1}{\prod_{n=0}^{N} \operatorname{vol}(b_n)} \frac{1}{\operatorname{vol}(b_{N+1})} \int_{B \times b_{N+1}} dx\, dy \\ &= \frac{1}{\prod_{n=0}^{N+1} \operatorname{vol}(b_n)} \int_B \int_{b_{N+1}} dy\, dx \end{aligned}$$

which equals $\mathrm{vol}_{(B)}/\prod_{n=0}^{N}\mathrm{vol}_{(b_n)}=\lambda_N(B)$. It follows from the Kolmogorov Extension of Measure Theorem (see, for example, Shiryayev [21, Theorem 3, and observation, p. 163]) that there exists a unique probability measure λ on D^r such that for every $B\subset D_N^r$

$$\lambda((c^0, ..., c^N, ...)\in D^r\colon (c^0, ..., c^N)\in B)=\lambda_N(B).$$

This uniform measure λ on D^r induces a uniform measure μ on $\mathscr{B}^r$, which will now be used to establish our main result.

4. MAIN RESULTS

Let $r>0$ and an integer $d\geqslant 1$ be given. Fix an integer $n\geqslant 1$, and set $\alpha(n)=c_1 n^{d/(d-1)}$, for some constant $c_1>0$ depending only on r and d. Let μ be the uniform measure over $\mathscr{B}^r$ constructed in Section 3.

THEOREM 1.

$$\mu\left\{f\in\mathscr{B}^r\colon \mathrm{dist}(f, M_n, L_2)\geqslant\frac{c_2}{n^{r/(d-1)}}\right\}\geqslant 1-e^{-\alpha(n)}$$

for some constant $c_2>0$ depending only on r and d.

THEOREM 2. *For all $f\in\mathscr{B}^r$*

$$\mathrm{dist}(f, M_n, L_2)\leqslant\frac{c_3}{n^{r/(d-1)}},$$

where $c_3>0$ is some constant depending only on r and d.

From Theorems 1 and 2 we have the following corollary which estimates the probabilistic width defined in (1).

COROLLARY 1. *Let $0\leqslant\delta<1-2e^{-\alpha(n)}$. Then*

$$\frac{c_2}{n^{r/(d-1)}}\leqslant d_{n,\delta}(\mathscr{B}^r, \mu, M_n)\leqslant\frac{c_3}{n^{r/(d-1)}}$$

for some constants $c_2, c_3>0$ depending only on r and d.

Indeed let $0\leqslant\delta<1-2e^{-\alpha(n)}$ be any number. Then for any set $A\subset\mathscr{B}^r$ with the measure $\mu(A)=1-\delta$ we have $\mu(A)\geqslant 2e^{-\alpha(n)}$. Therefore from

Theorem 1 it follows that there exists a function $f \in A$ such that $\operatorname{dist}(f, M_n, L_2) \geqslant c_2 n^{-r/(d-1)}$. Hence

$$d_{n,\delta}(\mathscr{B}^r, \mu, M_n) \geqslant \operatorname{dist}(f, M_n, L_2) \geqslant \frac{c_2}{n^{r/(d-1)}}.$$

The upper bound in Corollary 1 follows directly from Theorem 2.

We note that Traub *et al.* [25] consider also the so called average case setting which introduces the notion of an average distance with respect to a measure over a functional space in our case defined for $0 < p < \infty$ as

$$d_n^{avg}(\mathscr{B}^r, \mu, M_n)_p = \left(\int_{f \in \mathscr{B}^r} |\operatorname{dist}(f, M_n, L_2)|^p \, \mu(df) \right)^{1/p}.$$

The following corollary follows easily from Theorems 1 and 2.

COROLLARY 2. *For any* $0 < p < \infty$,

$$\frac{c_2}{n^{r/(d-1)}} \leqslant d_n^{avg}(\mathscr{B}^r, \mu, M_n)_p \leqslant \frac{c_3}{n^{r/(d-1)}}$$

for some constants $c_2, c_3 > 0$ *depending only on r, d, and p.*

We proceed to prove Theorem 1, first stating several auxiliary lemmas. From the definition of the orthonormal system $P = \{p_{s,l}\}$ it follows that an $h \in M_n$ can be expressed as a sum $\sum_{N=0}^{\infty} \sum_{(s,l) \in \Delta_N} c_{s,l}(h)\, p_{s,l}(x)$ with the coefficients, $c_{s,l}(h) = \langle h, p_{s,l} \rangle = \int_{B^d} h(x)\, p_{s,l}(x)\, dx$. Let $N \in \mathbb{Z}_+$ be some number, and $I \subset \Delta_N$ be any subset. Consider the set of sign-valued vectors

$$\Gamma_n^I := \{(\operatorname{sgn}(c_{s,l}(h))_{(s,l) \in I} : h \in M_n\}. \tag{5}$$

We will use the next lemma which follows from Lemma 3 of [12].

LEMMA 1. *Assume that N and n are such that* $|\Delta_N| = [c_5 n^{d/(d-1)}]$, *for some absolute constant* $c_5 > 0$. *Then for any subset* $I \subset \Delta_N$ *with* $|I| \geqslant |\Delta_N|/10$ *we have*

$$|\Gamma_n^I| \leqslant 2^{c_4 |I|} \leqslant 2^{c_6 n^{d/(d-1)}},$$

where $c_4 = 0.23$, *and* $c_6 = c_4 c_5$.

The next lemma then follows.

LEMMA 2. *Let* $|\Delta_N| = [c_5 n^{d/(d-1)}]$, *and let* $I \subset \Delta_N$, $|I| \geqslant |\Delta_N|/10$. *Introduce the sets of sign-valued vectors* $E^{|I|} = \{-1, +1\}^{|I|}$, *and* $\hat{E}^{|I|} = \{\varepsilon \in E^{|I|} : \operatorname{dist}(\varepsilon, \Gamma_n^I, l_2^{|I|}) \geqslant 2\sqrt{|I|/3}\}$. *Then*

$$|\hat{E}^{|I|}| \geqslant 2^{|I|} - 2^{c_7 |I|}$$

for some absolute constant $0 < c_7 < 1$.

Proof. Set $k = |I|$. From Lemma 1 it follows that the cardinality $|\Gamma_n^I| \leqslant 2^{c_4 k}$. Fix any $\varepsilon^* \in E^k$. Denote by

$$D_{\varepsilon^*} = \left\{\varepsilon \in E^k : \|\varepsilon - \varepsilon^*\|_{l_2^k}^2 \geqslant \frac{4k}{9}\right\}.$$

Now $|D_{\varepsilon^*}|$ is independent of the specific choice of $\varepsilon^* \in E^k$. As such $|D_{\varepsilon^*}| = |\{\varepsilon \in E^k : \|\varepsilon - \underline{1}\|_{l_2^k}^2 \geqslant 4k/9\}|$ where $\underline{1} = [1, ..., 1] \in E^k$. The latter equals $\sum_{i \geqslant k/9} \binom{k}{i}$ and is bounded from below by $2^k - 2^{c_8 k}$, $c_8 = 1 - 2(7/18)^2 \log_2 e = 0.55...$, where we used an upper bound on the tails of the binomial distribution (cf. [3]).

Set $\bar{D}_\varepsilon = E^k \backslash D_\varepsilon$. Then $|\bar{D}_{\varepsilon^*}| = |E^k \backslash D_{\varepsilon^*}| \leqslant 2^{c_8 k}$. We also have $\hat{E}^k := \bigcap_{\varepsilon^* \in \Gamma_n^I} D_{\varepsilon^*} = E^k \backslash (\bigcup_{\varepsilon^* \in \Gamma_n^I} \bar{D}_{\varepsilon^*})$. It follows that

$$|\hat{E}^k| \geqslant |E^k| - \left|\bigcup_{\varepsilon^* \in \Gamma_n^I} \bar{D}_{\varepsilon^*}\right| \geqslant |E^k| - |\Gamma_n^I| \, 2^{c_8 k} \geqslant 2^k - 2^{c_4 k} 2^{c_8 k}.$$

Set $c_7 = c_4 + c_8 = 0.78....$ Thus $|\hat{E}^k| \geqslant 2^k - 2^{c_7 k}$, which proves the lemma.

DEFINITION 1. Let B^m denote the unit ball in $\mathbb{R}^m$. For any set $A \subset B^m$ denote the volume of A as $\operatorname{vol}(A)$. The uniform measure over the ball denoted by ν is defined such that for every $A \subset B^m$, $\nu(A) = \operatorname{vol}(A)/\operatorname{vol}(B^m)$.

Denote by

$$A := \left\{x \in B^m : |x_k| > \frac{3}{8\sqrt{m}}, \text{ for at least } \frac{m}{10} \text{ coordinates } k\right\}.$$

We will use the following lemma.

LEMMA 3. *For any* $m \geqslant 1$

$$\nu(A) \geqslant 1 - 3e^{-c_9 m}$$

for some absolute constant $c_9 > 0$.

Proof. We aim at finding a lower bound on $\nu(A)$ by first expressing the measure of the set A under the uniform measure over B^m as the measure of another set under the Gaussian measure over $\mathbb{R}^m$. Introduce the auxiliary set in $\mathbb{R}^m$

$$\hat{A}=\left\{x \in \mathbb{R}^m : |x_k| > \frac{3}{4\sqrt{m}} \|x\|_2, \text{ for at least } \frac{m}{10} \text{ coordinates } k\right\}.$$

Denote by $B^m(\alpha, \beta) := B^m(\beta) \backslash B^m(\alpha)$. We have

$$\nu(A) \geqslant \nu(B^m(\tfrac{1}{2}, 1) \cap \hat{A}) \geqslant \nu(\hat{A} \cap B^m) - (\tfrac{1}{2})^m. \tag{6}$$

Let $\chi_A(x)$ denote the indicator function of the set A. Switching to polar coordinates we have, since $x \in \hat{A}$ implies $ax \in \hat{A}$ for all $a \neq 0$

$$\begin{aligned}\nu(\hat{A} \cap B^m) &= \frac{1}{\operatorname{vol}(B^m)} \int_{B^m} \chi_{\hat{A}}(x)\, dx \\ &= \frac{1}{\operatorname{vol}(B^m)} \int_0^1 r^{m-1}\, dr \int_{S^{m-1}} \chi_{\hat{A}}(s)\, ds \qquad (s \in S^{m-1}),\end{aligned} \tag{7}$$

where ds is the Lebesgue measure on S^{m-1}. Assume that m is even (for m odd the proof is analogous). The volume of the unit ball $\operatorname{vol}(B^m) = \pi^{m/2}/(m/2)!$. It is known (cf. [17]) that $\int_0^\infty x^{m-1} e^{-x^2}\, dx = \frac{1}{2}\Gamma(m/2)$. Hence it follows that

$$\frac{1}{\operatorname{vol}(B^m)} \int_0^1 r^{m-1}\, dr = \pi^{-m/2} \int_0^\infty r^{m-1} e^{-r^2}\, dr.$$

Therefore using once more polar coordinates we obtain from (7)

$$\begin{aligned}\nu(\hat{A} \cap B^m) &= \pi^{-m/2} \int_0^\infty r^{m-1} e^{-r^2}\, dr \int_{S^{m-1}} \chi_{\hat{A}}(s)\, ds \\ &= \pi^{-m/2} \int_{\mathbb{R}^m} \chi_{\hat{A}}(x)\, e^{-|x|^2}\, dx.\end{aligned} \tag{8}$$

Define a Gaussian measure over $\mathbb{R}^m$ as $\gamma(G) = \pi^{-m/2} \int_G e^{-|x|^2}\, dx$, $G \subset \mathbb{R}^m$. From (8) it is seen that $\nu(\hat{A} \cap B^m) = \gamma(\hat{A})$. Let

$$D=\left\{x \in \mathbb{R}^m : |x_k| \geqslant \frac{3}{2}, \text{ for at least } \frac{m}{10} \text{ coordinates } k\right\}.$$

Then it follows that

$$\nu(\hat{A} \cap B^m) = \gamma(\hat{A}) \geqslant \gamma(\hat{A} \cap B^m(2\sqrt{m})) \geqslant \gamma(D \cap B^m(2\sqrt{m})),$$

and therefore

$$\nu(\hat{A} \cap B^m) \geqslant \gamma(D) + \gamma(B^m(2\sqrt{m})) - \gamma(D \cup B^m(2\sqrt{m}))$$
$$\geqslant \gamma(D) + \gamma(B^m(2\sqrt{m})) - 1. \tag{9}$$

Let $I \subset \mathbb{Z}_m = \{1, 2, ..., m\}$. Consider the subset in D

$$D_I = \{x \in D: |x_i| \geqslant \tfrac{3}{2} \text{ for all } i \in I, |x_i| < \tfrac{3}{2} \text{ for all } i \in \mathbb{Z}_m \backslash I\}.$$

We have

$$\gamma(D) = \sum_{I \subset \mathbb{Z}_m} \gamma(D_I) = \sum_{l=1}^{m} \sum_{I \subset \mathbb{Z}_m, |I| = l} \gamma(D_I) \geqslant \sum_{l=m/10}^{m} \sum_{I \subset \mathbb{Z}_m, |I| = l} \gamma(D_I).$$

For $|I| = l$

$$\gamma(D_I) = p^l(1-p)^{m-l},$$

where

$$\frac{1}{\sqrt{\pi}} \int_{|t| \geqslant 3/2} e^{-t^2} \, dt = 0.134 \equiv p.$$

Hence from the definition of the Gaussian measure γ it follows that

$$\gamma(D) \geqslant \sum_{l=m/10}^{m} \binom{m}{l} p^l(1-p)^{m-l} > 1 - e^{-c_{10} m} \tag{10}$$

for some $0 < c_{10} < 1$ where we used a bound on the tail of the binomial distribution [3].

We now estimate $\gamma(B^m(2\sqrt{m}))$. We will show that

$$\gamma(B^m(2\sqrt{m})) \geqslant 1 - e^{-c_{11} m} \tag{11}$$

for some absolute constant $c_{11} > 0$.

Indeed using polar coordinates we have

$$\gamma(B^m(2\sqrt{m})) = \pi^{-m/2} \int_{B^m(2\sqrt{m})} e^{-|x|^2} \, dx$$
$$= 1 - \pi^{-m/2} \int_{\mathbb{R}^m \backslash B^m(2\sqrt{m})} e^{-|x|^2} \, dx$$
$$= 1 - \pi^{-m/2} \, d(S^{d-1}) \int_{2\sqrt{m}}^{\infty} r^{m-1} e^{-r^2} \, dr,$$

where $d(S^{d-1})$ is the Lebesgue measure of the sphere S^{d-1}. Using the substitution $r=\sqrt{mt/2}$, and the estimate $\int_8^\infty t^{k-1}e^{-kt}\,dr \leqslant (1/7k)\,e^{-8k}8^{k+1/2}$, $k\geqslant 1$ (see [10, p. 471, form. (6.5)]), we obtain

$$\int_{2\sqrt{m}}^{\infty} r^{m-1}e^{-r^2}\,dr = \tfrac{1}{2}\,(m/2)^{m/2}\int_8^{\infty} t^{m/2-1}e^{-(m/2)\,t}\,dt \leqslant \tfrac{1}{2}(m/2)^{m/2}\,e^{-c'_{11}m},$$

where $c'_{11}=4-\frac{3}{2}\ln 2$. Since $d(S^{d-1})=\mathrm{vol}(B^m)/m=\pi^{m/2}/m\Gamma(m/2)\asymp\pi^{m/2}\times e^{m/2}/(m(m/2)^{m/2}\sqrt{2\pi m})$, then

$$\gamma(B^m(2\sqrt{m}))\geqslant 1-\pi^{-m/2}\,d(S^{d-1})\,\tfrac{1}{2}(m/2)^{m/2}\,e^{-c'_{11}m}\geqslant 1-e^{-c_{11}m},$$

where $c_{11}=\frac{3}{2}(1-\ln 2)$.

Using (6), (9), (10), and (11) we obtain that

$$\nu(A)\geqslant 1-e^{-c_{10}m}-e^{-c_{11}m}-2^{-m}\geqslant 1-3e^{-c_9 m},$$

for absolute constant $c_9=\min\{c_{10}, c_{11}, \ln 2\}$. ∎

We now proceed with finding a lower bound on the measure stated in Theorem 1.

4.1. *Proof of Theorem* 1

The proof of Theorem 1 is based on the following observation. Let $m=|\Delta_N|$. In the space $\mathbb{R}^m$, consider the set $E^m=\{-1,+1\}^m$ endowed with a uniform discrete measure α, and let $\Gamma_n^{\Delta_N}$ be the subset in E^m defined in (5). From Lemma 2 it follows that the measure of elements in E^m which are "badly" approximated by the manifold $\Gamma_n^{\Delta_N}$, i.e., the α measure of set $G=\{\varepsilon\in E^m\colon \mathrm{dist}(\varepsilon, \Gamma_n^{\Delta_N}, l_2^m)\geqslant 2\sqrt{m/3}\}$ satisfies the inequality

$$\alpha(G)\geqslant 1-2^{-cm}$$

for $c>0$. This implies that almost all elements from E^m, in the sense of the induced probabilistic measure over E^m, are "badly" approximated by $\Gamma_n^{\Delta_N}$. The statement of the theorem follows upon making use of the isomorphism (4).

We proceed with the detailed proof. Let $N^*>0$ be some integer which will be taken later to be sufficiently large. Since

$$\mathrm{dist}(f, M_n, L_2)^2=\inf_{h\in M_n}\sum_{N=0}^{\infty}\sum_{(s,\,l)\in\Delta_N}|c_{s,\,l}(f)-c_{s,\,l}(h)|^2$$

$$\geqslant \inf_{h\in M_n}\sum_{(s,\,l)\in\Delta_{N^*}}|c_{s,\,l}(f)-c_{s,\,l}(h)|^2,$$

then for an arbitrary $\varepsilon > 0$

$$\mu\{f \in \mathscr{B}^r : \operatorname{dist}(f, M_n, L_2) > \varepsilon\}$$

$$\geqslant \mu \left\{ f \in \mathscr{B}^r : \inf_{h \in M_n} \sum_{(s,l) \in \Delta_{N^*}} |c_{s,l}(f) - c_{s,l}(h)|^2 > \varepsilon^2 \right\}. \tag{12}$$

Let m, N^* and n be such that $m = |\Delta_{N^*}| = c_5 n^{d/(d-1)}$. To any $h \in M_n$ there corresponds a vector $\hat{h} \in \mathbb{R}^m$ defined as

$$\hat{h} = (c_{s,l}(h))_{(s,l) \in \Delta_{N^*}}. \tag{13}$$

Denote by

$$\hat{M}_n = \{\hat{h} = (\hat{h}_1, ..., \hat{h}_m) \in \mathbb{R}^m : h \in M_n\}.$$

Due to the isomorphism statement of (4) the approximation problem is now reduced to approximation in an m-dimensional Euclidean space.

Let $\varepsilon = 2^{rN^*}/4$ in (12). We have

$$\Sigma := \mu \left\{ f \in \mathscr{B}^r : \inf_{h \in M_n} \sum_{k \in \Delta_{N^*}} |c_k(f) - c_k(h)|^2 > \varepsilon^2 \right\}$$

$$= \nu \left\{ y \in B^m : \inf_{\hat{h} \in \hat{M}_n} \sum_{i=1}^{m} |y_i - \hat{h}_i|^2 > \tfrac{1}{4} \right\}.$$

Let $I \subseteq \mathbb{Z}_m$. Define the set

$$Q_I = \left\{ x \in B^m : |x_i| \geqslant \frac{3}{8\sqrt{m}}, \text{ for all } i \in I, |x_i| \leqslant \frac{3}{8\sqrt{m}} \text{ for all } i \in \mathbb{Z}_m \backslash I \right\}.$$

From the definition of Q_I we have $\bigcup_{I \in \mathbb{Z}_m} Q_I = B^m$. Thus

$$\Sigma = \sum_{I \in \mathbb{Z}_m} \nu \left\{ y \in Q_I : \inf_{\hat{h} \in \hat{M}_n} \sum_{i=1}^{m} |y_i - \hat{h}_i|^2 > \tfrac{1}{4} \right\}.$$

For all $I \subset \mathbb{Z}_m$, $|I| \geqslant m/10$, and $y \in Q_I$ we have

$$\sum_{i=1}^{m} |y_i - \hat{h}_i|^2 \geqslant \sum_{i \in I} |y_i - \hat{h}_i|^2 \geqslant \frac{9}{64m} \sum_{i \in I} \left| \frac{y_i}{|y_i|} - \frac{\hat{h}_i}{|y_i|} \right|^2.$$

Denote by $\varepsilon_i(y) = y_i/|y_i| = \operatorname{sgn}(y_i)$. Then using the fact that for any $a \in \mathbb{R}$ and $\delta \in \{-1, +1\}$ the inequality $|\delta - a| \geqslant \frac{1}{2}|\delta - \operatorname{sgn}(a)|$ holds we have

$$\sum_{i=1}^{m} |y_i - \hat{h}_i|^2 \geqslant \frac{9}{256m} \sum_{i \in I} |\varepsilon_i(y) - \operatorname{sgn}(\hat{h}_i)|^2.$$

We then have for $b = 64/9$

$$\begin{aligned} \Sigma &\geqslant \sum_{I \in \mathbb{Z}_m:\, |I| \geqslant m/10} \nu \left\{ y \in Q_I : \inf_{\hat{h} \in \hat{M}_n} \sum_{i \in I} |\varepsilon_i(y) - \operatorname{sgn}(\hat{h}_i)|^2 > bm \right\} \\ &= \sum_{j=0}^{9m/10} \sum_{I \in \mathbb{Z}_m:\, |I| = m/10 + j} \nu \left\{ y \in Q_I : \inf_{\hat{h} \in \hat{M}_n} \sum_{i \in I} |\varepsilon_i(y) - \operatorname{sgn}(\hat{h}_i)|^2 > bm \right\}. \end{aligned}$$

For $I \in \mathbb{Z}_m$ let $E^{|I|} = \{-1, +1\}^{|I|}$. Define

$$\Gamma_n^{|I|} = \{ (\operatorname{sgn}(\hat{h}_i))_{i \in I} : h \in M_n \}.$$

Denote by $\|y\|_{l_2^{|I|}} = (\sum_{i \in I} |y_i|^2)^{1/2}$. Let

$$\hat{E}^{|I|} = \{ \varepsilon \in E^{|I|} : \min_{\delta \in \Gamma_n^{|I|}} \|\varepsilon - \delta\|^2_{l_2^{|I|}} \geqslant bm \}. \tag{14}$$

For any $\varepsilon = (\varepsilon_i)_{i \in I} \in E^{|I|}$ define the set

$$Q_{I,\varepsilon} = \{ y \in Q_I : \operatorname{sgn}(y_i) = \varepsilon_i, \text{ for all } i \in I \}.$$

Then continuing from above we have

$$\begin{aligned} \Sigma &\geqslant \sum_{j=0}^{9m/10} \sum_{I \in \mathbb{Z}_m:\, |I| = m/10 + j} \nu \{ y \in Q_I : \min_{\delta \in \Gamma_n^{|I|}} \|\varepsilon(y) - \delta\|^2_{l_2^{|I|}} > bm \} \\ &= \sum_{j=0}^{9m/10} \sum_{I \in \mathbb{Z}_m:\, |I| = m/10 + j} \sum_{\varepsilon \in E^{|I|}} \nu \{ y \in Q_{I,\varepsilon} : \min_{\delta \in \Gamma_n^{|I|}} \|\varepsilon(y) - \delta\|^2_{l_2^{|I|}} > bm \} \end{aligned}$$

and since $\hat{E}^{|I|} \subset E^{|I|}$ then

$$\Sigma \geqslant \sum_{j=0}^{9m/10} \sum_{I \in \mathbb{Z}_m:\, |I| = m/10 + j} \sum_{\varepsilon \in \hat{E}^{|I|}} \nu \{ y \in Q_{I,\varepsilon} : \min_{\delta \in \Gamma_n^{|I|}} \|\varepsilon(y) - \delta\|^2_{l_2^{|I|}} > bm$$

Now from (14) for all $\varepsilon \in \hat{E}^{|I|}$ the condition $\min_{\delta \in \Gamma^I} \|\varepsilon - \delta\|^2_{l_2^{|I|}} > bm$ is satisfied. We therefore have

$$\Sigma \geqslant \sum_{j=0}^{9m/10} \sum_{I \in \mathbb{Z}_m:\, |I| = m/10 + j} \sum_{\varepsilon \in \hat{E}^{|I|}} \nu \{ y \in Q_{I,\varepsilon} \}.$$

Note that $\nu(y \in Q_{I,\varepsilon})$ does not depend on ε. Denote by $a_I := \nu\{y \in Q_{I,\varepsilon}\}$. Thus the latter becomes

$$\sum_{j=0}^{9m/10} \sum_{I \in \mathbb{Z}_m : |I| = m/10 + j} \sum_{\varepsilon \in \hat{E}^{|I|}} a_I = \sum_{j=0}^{9m/10} \sum_{I \in \mathbb{Z}_m : |I| = m/10 + j} a_I \, |\hat{E}^{|I|}|.$$

From Lemma 2 it follows that for I such that $|I| = m/10 + j$ the cardinality $|\hat{E}^{|I|}| \geqslant 2^{m/10+j} - 2^{c_7(m/10+j)}$ for some constant $0 < c_7 < 1$. We therefore have

$$\Sigma \geqslant \sum_{j=0}^{9m/10} \sum_{I \in \mathbb{Z}_m : |I| = m/10 + j} a_I 2^{m/10+j} (1 - 2^{-(1-c_7)(m/10+j)})$$

$$\geqslant (1 - 2^{-(1-c_7)\, m/10}) \sum_{j=0}^{9m/10} \sum_{I \in \mathbb{Z}_m : |I| = m/10 + j} a_I 2^{m/10+j}.$$

Since $a_I 2^{m/10+j} = |E^{|I|}| \, a_I = \nu(Q_I)$ then

$$\Sigma \geqslant (1 - 2^{-(1-c_7)\, m/10})$$

$$\times \nu \left\{ y \in B^m : |y_k| > \frac{3}{8\sqrt{m}}, \text{ for at least } \frac{m}{10} \text{ coordinates } k \right\}.$$

Using Lemma 3 we have

$$\Sigma \geqslant (1 - 2^{-(1-c_7)\, m/10})(1 - 3e^{-c_9 m}) \geqslant 1 - e^{-c_{12} m}$$

for some absolute constants c_9, $c_{12} > 0$. Finally, from before, $m = 2^{dN^*}$ and $\rho = 2^{-rN^*}$ then $\rho \asymp m^{-r/d}$. Also, the condition of Lemma 1 has $m = c_5 n^{d/(d-1)}$ thus $\rho = c_{13}/n^{r/(d-1)}$ and therefore

$$\mu \left\{ f \in \mathscr{B}^r : \operatorname{dist}(f, M_n, L_2) > \frac{c_{13}}{4n^{r/(d-1)}} \right\} \geqslant 1 - e^{-\alpha(n)},$$

where $\alpha(n) = c_5 c_{11} n^{d/(d-1)}$. This completes the proof of Theorem 1. ∎

4.2. *Proof of Theorem 2*

Let $\mathscr{P}_s$ and $\mathscr{P}_s^h$ be as defined in Section 2. Choose n such that $n = \dim(\mathscr{P}_s^h)$. Then from Proposition 2 of [12] it follows that $\mathscr{P}_s \subset M_n$. Let N' be the integer such that $2^{N'-1} \leqslant s \leqslant 2^{N'}$. Considering the definition of $\mathscr{B}^r$ we have for all $f \in \mathscr{B}^r$, $f(x) = \sum_{N=0}^{\infty} \sum_{(s,l) \in \Delta_N} c_{s,l} p_{s,l}(x)$ and

$$\operatorname{dist}(f, M_n, L_2)^2 \leqslant \operatorname{dist}(f, \mathscr{P}_s, L^2)^2 \leqslant \left\| \sum_{N \geqslant N'} \sum_{(s,l) \in \Delta_N} c_{s,l} p_{s,l} \right\|_{L_2}^2.$$

Therefore from the Parseval equality and the definition of class $\mathscr{B}^r$ we obtain

$$\operatorname{dist}(f, M_n, L_2)^2 \leqslant \sum_{N \geqslant N'} \left\| \sum_{(s,\, l) \in \Delta_N} c_{s,\, l} p_{s,\, l} \right\|_{L_2}^2$$

$$\leqslant \sum_{N \geqslant N'} 2^{-2rN} \leqslant c_{14} 2^{-2rN'} = c_{14} s^{-2r},$$

for some constant $c_{14} > 0$. It is known (cf. [22]) that $\dim(\mathscr{P}_s^h) = \binom{d+s-1}{d-1} \asymp s^{d-1}$. Since $n \asymp s^{d-1}$ then $s \asymp n^{1/(d-1)}$. Thus

$$\operatorname{dist}(f, M_n, L_2) \leqslant c_{15} n^{-r/(d-1)},$$

which proves the theorem.

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