The set covering polyhedron of circular matrices: Minor vs. row family inequalities*

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Abstract. In the context of the study of the set covering polyhedron of circular matrices, we study the relationship between row family inequalities and a previously proposed class termed minor inequalities. Recently, a complete linear description of this polyhedron has been provided in terms of row family inequalities. In this work we prove that for the particular subclass of circulant matrices, the corresponding row family inequalities are related to circulant minors. Moreover, we extend previous results on circulant matrices C_{sk}^k, $s \in \{2,3\}$ and $k \geq 2$, by providing a polynomial time separation algorithm for the inequalities describing the set covering polyhedron of matrices C_{4k}^k, $k \geq 2$.

Keywords: circular matrix · set covering polyhedron · circulant minor

1 Introduction

The (weighted) *set packing problem* (SPP) and *set covering problem* (SCP) are classic problems in combinatorial optimization with important practical applications, but hard to solve in general. These problems can be stated as

$$\begin{array}{ll}(\text{SPP}) & \max\{c^T x : Ax \leq \mathbf{1}, x \in \{0,1\}^n\},\\ (\text{SCP}) & \min\{c^T x : Ax \geq \mathbf{1}, x \in \mathbb{Z}_+^n\},\end{array}$$

where A is an $m \times n$ matrix with $0, 1$ entries, $c \in \mathbb{Z}^n$, and $\mathbf{1} \in \mathbb{Z}^m$ is the vector having all entries equal to one.

One established approach to tackle these problems has been to study the polyhedral properties of their sets of feasible solutions. The *set packing polytope* $P^*(A)$ (resp. *set covering polyhedron* $Q^*(A)$) is defined as the convex hull of all feasible solutions of SPP (resp. SCP). The structure of $P^*(A)$ and $Q^*(A)$ has been extensively addressed in previous works (see, e.g., [10, 11] for two seminal

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articles). However, the set packing problem has traditionally received more attention in the past, mainly in the context of the equivalent stable set problem on graphs.

Although SPP and SCP are NP-hard problems in general, some interesting polynomial time solvable particular cases have been described in the literature. It is natural to ask whether an explicit description in terms of linear inequalities can be provided for $P^*(A)$ or $Q^*(A)$ in these cases. In this paper, we focus on the case of circular matrices and their subclass of circulant matrices defined in the next section. The set packing polytope related to these matrices has been completely described in [8, 12] and more recent works [2–5, 13, 14] have addressed the task of obtaining a similar results for the set covering polyhedron.

The characterization of the set covering polyhedron related to circular matrices has followed two lines of research. On one hand, the class of *minor inequalities* was proposed in [2] and further studied and extended in subsequent works. These valid inequalities are related to certain substructures of the circular matrix called circulant minors. On the other hand, *row family inequalities* were introduced in [3] as a counterpart of *clique family inequalities*, which played a central role in the linear description of the set packing polytope of circular matrices in [8]. Recently, a complete linear description of $Q^*(A)$ has indeed been provided in terms of row family inequalities [14]. Moreover, minor inequalities can be interpreted as a special case of row family inequalities. In this paper, we further explore the relationship between these two classes of inequalities. As it turns out, certain row family inequalities related to circulant minors are sufficient for describing the set covering polyhedron related to circulant matrices.

From an algorithmic point of view, it is important to ask whether these new classes of valid (or facet defining) inequalities can be separated efficiently. The separation problem for certain classes of minor inequalities has been addressed in previous works [4, 13]. In this article, we provide a new polynomial separation algorithm for a class of row family inequalities related to certain relevant circulant matrices.

The paper is organized as follows: in the next section we introduce some notation and preliminary concepts, among them the classes of minor and row family inequalities for the set covering polyhedron. In Section 3 a combinatorial interpretation of row family inequalities is provided. Based on this interpretation, a complete linear description of the set covering polyhedron related to circulant matrices is proposed in Section 4, in terms of a particular subclass of row family inequalities related to circulant minors. Finally, a separation algorithm for a new class of row family inequalities in presented in Section 5.

2 Preliminaries

For $n \in \mathbb{N}$, let $[n]$ denote the additive group defined on the set $\{1, \ldots, n\}$, with integer addition modulo n. Given A a $0,1$ matrix we say that a row v of A is a dominating row if $v \geq u$ for some u row of A with $u \neq v$. Throughout this article, we consider matrices with $0,1$ entries, without zero columns and without dominating rows. Moreover, if A is such a matrix of order $m\times n$, then we consider the columns (resp. rows) of A to be indexed by $[n]$ (resp. by $[m]$). Two matrices A and A' are *isomorphic*, denoted by $A \approx A'$, if A' can be obtained from A by permutation of rows and columns.

A *cover* of A is a subset of $[n]$ whose incidence vector x satisfies $Ax \geq \mathbf{1}$. Whenever there is no risk of confusion, we refer indistinctly to a subset of $[n]$ and to its incidence vector. The *covering number* $\tau(A)$ of a matrix A is the minimum cardinality of a cover of A.

The term *boolean inequality* denotes each of the inequalities of the system $Ax \geq \mathbf{1}$, $x \geq \mathbf{0}$. The inequality $\sum_{i=1}^{n} x_i \geq \tau(A)$ is called the *rank constraint*, and it is always valid for $Q^*(A)$.

A matrix A is called *circular* if, for every row $i \in [m]$, there are two integer numbers $\ell_i, k_i \in [n]$ and $2 \leq k_i \leq n-1$ such that the i-th row of A is the incidence vector of the set $C^i := \{\ell_i, \ell_i+1, \ldots, \ell_i+(k_i-1)\} \subset [n]$. In the special case where A is a square circular matrix of order n and $\ell_i = i$, $k_i = k$ hold for every row $i \in [n]$, A is called a *circulant matrix* and denoted by C_n^k.

When $A = C_n^k$ it is known that $\tau(C_n^k) = \lceil \frac{n}{k} \rceil$ and the rank constraint is a facet of $Q^*(C_n^k)$ if and only if n is not a multiple of k [11].

Given $N \subset [n]$, the *minor of A obtained by contraction of N*, denoted by A/N, is the submatrix of A that results after removing all columns with indices in N and all dominating rows. In this work, anytime we refer to a minor of a matrix, we mean a minor obtained by contraction.

A minor of a matrix A is called a *circulant minor* if it is isomorphic to a circulant matrix.

In [8, 12] the authors proposed a complete linear description of the stable set polytope of circular graphs, which is equivalent to obtaining a complete linear description for the set packing polytope related to circular matrices. The authors show that if A is a circular matrix then $P^*(A)$ is completely described by three classes of inequalities: (i) nonnegativity constraints, (ii) clique inequalities, and (iii) the class of *clique family inequalities* introduced in [9].

Following a similar pattern, in [3] the class of *row family inequalities* (rfi) was proposed as a counterpart, in the set covering case, of clique family inequalities. We describe them at next, slightly modified to fit in our current notation.

Let $A = (a_{ij})_{m\times n}$, $F \subset [m]$ a set of row indices of A, with $s := |F| \geq 2$, $p \in [s-1]$ such that s is not a multiple of p, and $r := s - p \left\lfloor \frac{s}{p} \right\rfloor$. Define the sets

$$I(F,p) = \left\{ j \in [n] \,:\, \sum_{i\in F} a_{ij} \leq p \right\}, \quad O(F,p) = \left\{ j \in [n] \,:\, \sum_{i\in F} a_{ij} = p+1 \right\}.$$

The *row family inequality* (rfi) *induced by* (F,p) is

$$(r+1)\sum_{j\in O(F,p)} x_j \,+ r\sum_{j\in I(F,p)} x_j \geq r \left\lceil \frac{s}{p} \right\rceil . \tag{1}$$

Row family inequalities generalize several previously known classes of valid inequalities for $Q^*(A)$. However, in contrast to clique family inequalities, row family inequalities are not valid for $Q^*(A)$ in general. In [3] it is proved that (1) is valid for $Q^*(A)$ if the following condition holds for every cover B of A:

$$p\,|B \cap I(F,p)| + (p+1)\,|B \cap O(F,p)| \geq s. \tag{2}$$

In particular, if $p + 1 = \max_{j\in[n]} \sum_{i\in F} a_{ij}$, the inequality is always valid for $Q^*(A)$. More recently, following the same ideas proposed in [8] for the set packing case, it has been proved that

Theorem 1. *[14] Given a circular matrix $A \in \{0,1\}^{m\times n}$, every non boolean non rank facet defining inequality of $Q^*(A)$ is a row family inequality induced by (F,p) with $F \subset [m]$ and $p+1 = \max_{j\in[n]} \sum_{i\in F} a_{ij}$. Moreover, if $s = |F|$, the roots of this facet defining inequality have cardinality $\left\lceil \frac{s}{p} \right\rceil$ or $\left\lceil \frac{s}{p} \right\rceil - 1$.*

In the particular case when $A = C_n^k$, non rank facet defining inequalities of $Q^*(C_n^k)$ related to circulant minors were studied in [2, 4, 5, 13, 14]. Given $N \subset [n]$ such that $C_n^k/N \approx C_{n'}^{k'}$, let $W := \{j \in N : j-k-1 \in N\}$. Then, the inequality

$$2\sum_{j\in W} x_j + \sum_{j\notin W} x_j \geq \left\lceil \frac{n'}{k'} \right\rceil \tag{3}$$

is valid for $Q^*(C_n^k)$, and facet defining if $n' \mod k' = 1$. This inequality is termed as the *minor inequality* induced by N [2].

Circulant minors, and the inequalities related to them, have an interesting combinatorial characterization in terms of circuits in a particular digraph. In fact, given a circulant matrix C_n^k, a directed auxiliary graph $G(C_n^k)$ is defined in [7], by considering n nodes and arcs of the form $(i, i+k)$ and $(i, i+k+1)$ for every $i \in [n]$. The authors prove that if $N \subset [n]$ induces a simple circuit in $G(C_n^k)$, then the matrix C_n^k/N is a circulant minor of C_n^k. Later, Aguilera [1] showed that C_n^k/N is isomorphic to a circulant minor of C_n^k if and only if N induces $d \geq 1$ disjoint simple circuits in $G(C_n^k)$, each one having the same number of arcs of length k and $k+1$.

Using this last result, it has been proved in [4] that if $C_n^k/N \approx C_{n'}^{k'}$ holds for some $N \subset [n]$, and if $F := \{i \in [n] : i+1 \notin N\}$, then $k'+1 = \max_{j\in[n]} \sum_{i\in F} a_{ij}$, $n' = |F|$, and $\{j \in [n] : \sum_{i\in F} a_{ij} = k'+1\} = \{j : j-k-1 \in N\} = W$. Moreover, the rfi induced by (F, k') has the form

$$(r+1)\sum_{j\in W} x_j + r\sum_{j\notin W} x_j \geq r\left\lceil \frac{n'}{k'} \right\rceil, \tag{4}$$

with $r = n' - k'\left\lfloor \frac{n'}{k'} \right\rfloor$. These inequalities generalize minor inequalities (3), as the latter ones correspond to the case when $r = 1$. Since, at the time, they are a particular subclass of row family inequalities, they are termed as *minor related row family inequalities* in [14].

In the next section we provide a combinatorial characterization of the rfi's described in Theorem 1. Our result is obtained by following the same ideas proposed in [8, 12] for a similar characterization of the clique family inequalities related to the stable set polytope of circular graphs.

3 A combinatorial characterization of row family inequalities

Given a circular matrix A, let $D(A) = (V, E)$ be a directed graph where $V = [n]$ and the arcs in E are of the form:

(i) $\{(l_i - 1, l_i + k_i - 1) : i \in [m]\}$, called *row arcs*. Observe that each row arc corresponds to a row of matrix A.
(ii) $\{(i - 1, i) : i \in [n]\}$ and $\{(i, i - 1) : i \in [n]\}$, called $(+1)$-arcs and (-1)-arcs, respectively.

From the proof of Theorem 1, it follows that each rfi describing a facet of $Q^*(A)$ is induced by the set of rows corresponding to the row arcs of a simple circuit in $D(A)$. In the following, we characterize the structure of the rfi's describing $Q^*(A)$ in terms of the combinatorial parameters of its associated simple circuits in $D(A)$.

Given a simple circuit Γ in $D(A)$, we denote by $V(\Gamma)$ and $E(\Gamma)$ the set of nodes and the set of arcs in Γ. A simple circuit Γ in $D(A)$ partitions the set of nodes V into the following sets:

(i) $\circ(\Gamma) = \{i \in [n] : (i - 1, i) \in \Gamma\}$, denoted as the *circles* of Γ,
(ii) $\otimes(\Gamma) = \{i \in [n] : (i, i - 1) \in \Gamma\}$, called the *crosses* of Γ, and,
(iii) $\bullet(\Gamma) = [n] \setminus (\circ(\Gamma) \cup \otimes(\Gamma))$, the *bullets* of Γ.

Observe that if $i \in \bullet(\Gamma)$ then either $i \notin V(\Gamma)$ or it is the tail or the head of a row arc in Γ.

Given $e \in E$, we denote by $l(e)$ the length of e, defined as follows: if e is a row arc corresponding to the i-th row of A, $l(e) = k_i$, if e is a $(+1)$-arc (resp. (-1)-arc), $l(e) = 1$ (resp. $l(e) = -1$).

The *winding number* of a simple circuit Γ in D is

$$\frac{\sum_{e \in E(\Gamma)} l(e)}{n}.$$

Circuits in $D(A)$ fulfill the following property:

Theorem 2. *Let A be a circular matrix and Γ a simple circuit in $D(A)$ with winding number p. Let $F = \{i \in [m] : (l_i - 1, l_i + k_i - 1) \in E(\Gamma)\}$. Then the following statements hold:*

(i) if $i \in \circ(\Gamma)$ then i belongs to exactly $p - 1$ rows of the set $\{C^j : j \in F\}$,
(ii) if $i \in \bullet(\Gamma)$ then i belongs to exactly p rows of the set $\{C^j : j \in F\}$,
(iii) $i \in \otimes(\Gamma)$ then i belongs to exactly $p + 1$ rows of the set $\{C^j : j \in F\}$.

From the previous result, given a simple circuit Γ in $D(A)$, if we define $F := \{i : (l_i - 1, l_i + k_i - 1) \in \Gamma\}$ and $p := \max\{\sum_{i \in F} a_{ij} : j \in [n]\} - 1$, then p is the winding number of Γ and the rfi induced by (F, p) can be written as

$$(r + 1) \sum_{i \in \otimes(\Gamma)} x_i + r \sum_{i \in \circ(\Gamma) \cup \bullet(\Gamma)} x_i \geq r \left\lceil \frac{s}{p} \right\rceil, \tag{5}$$

where $s = |F|$ and $r = s - p \left\lfloor \frac{s}{p} \right\rfloor$.

Given a simple circuit Γ in $D(A)$, we call (5) *the inequality associated with* Γ.

The next result relates facets of $Q^*(A)$ with circuits in $D(A)$.

Theorem 3. *Given a circular matrix A, every non boolean non rank facet defining inequality of $Q^*(A)$ is associated with a simple circuit Γ in $D(A)$ with at least 5 row arcs and winding number at least 2.*

4 Facet defining inequalities and circulant minors

Given a circular matrix A, in this section we will associate with every facet defining inequality of $Q^*(A)$ a circulant submatrix of A. The results are strongly related to the ideas in [12] for the stable set polytope of circular graphs.

In the particular case when A itself is a circulant matrix, we prove that this submatrix is a minor, and then, all facet defining rfi's belong to the class of minor related row family inequalities (4).

Given a simple circuit Γ in $D(A)$, we say that a node is essential (w.r.t. Γ) if it is a bullet or a cross and it is the head or the tail of a row arc in Γ.

Theorem 4. *Let A be a circular matrix, Γ a simple circuit in $D(A)$, and $F = \{i : (l_i - 1, l_i + k_i - 1) \in \Gamma\}$. Then, the submatrix of A consisting of the rows indexed by F and the columns indexed by the essential bullets of Γ is isomorphic to the circulant C_s^p, where $s := |F|$ and p is the winding number of Γ. Moreover,* $\gcd(s, p) = 1$.

The previous result does not necessarily imply that the circulant submatrix C_s^p is a minor of A, as some of its rows may dominate other rows of A (when restricted to the columns indexed by the essential bullets of Γ). However, it is not hard to see that this is the case when the set of essential nodes coincides with $[n]$. Moreover, as we shall see at next, this is also the case when the matrix A is circulant. Indeed, circulant matrices fulfill the following stronger property.

Theorem 5. *Let C_n^k be a circulant matrix, and Γ a simple circuit in $D(C_n^k)$ such that its associated inequality defines a facet of $Q^*(C_n^k)$. Then, $\circ(\Gamma) = \emptyset$.*

Remark 1. Observe that, from the previous theorem, facet defining inequalities of $Q^*(C_n^k)$ are associated with simple circuits in $D(C_n^k)$ without $(+1)$-arcs.

This property is crucial to prove the next result.

Theorem 6. *Let C_n^k be a circulant matrix and Γ a simple circuit in $D(C_n^k)$ such that its associated row family inequality defines a facet of $Q^*(C_n^k)$. Then, the circulant submatrix of C_n^k consisting of the rows indexed by F and the columns indexed by the essential bullets of Γ is a circulant minor of C_n^k. Moreover, the inequality associated with Γ coincides with (4), where $s = |F|$, p is the winding number of Γ, and $gcd(s, p) = 1$.*

Summarizing all the results obtained so far we have:

Theorem 7. *For any circulant matrix C_n^k, $Q^*(C_n^k)$ is completely described by:*

- *boolean constraints,*
- *the rank constraint,*
- *minor related row family inequalities of the form* (4)*, induced by circulant minors C_s^p with s and p relative prime numbers.*

Theorem 7 can be seen as the counterpart for the set covering polyhedron of circulant matrices of a result obtained by Stauffer [12] for the stable set polytope of web graphs $\mathrm{STAB}(W_n^k)$. His result states that every non boolean non rank facet defining inequality of $\mathrm{STAB}(W_n^k)$ is related to a prime subweb.

5 Polynomial separation routines for SCP on matrices C_{tk}^k, with $t = 2, 3, 4$

In [5] the authors prove that matrices C_n^k with $n = tk$ and $t \geq 2$ play an important role for the study of the set covering polyhedron of general circulant matrices. Indeed, given a fixed $k \in \mathbb{N}$, there exists $\hat{t} \geq 2$ such that C_n^k is a *minor* of $C_{\hat{t}k}^k$ for almost all values of n (i.e., except for a finite number of values of n). Thus, for almost all values of n, a description of the set covering polyhedron $Q^*(C_n^k)$ can be obtained from a description of the set covering polyhedron $Q^*(C_{\hat{t}k}^k)$.

In the same work, the authors prove that, for all $t \geq 2$, facet defining inequalities of $Q^*(C_{tk}^k)$ with a right hand side being a multiple of $t+1$ (called $(t+1)$-inequalities) can be separated in polynomial time by solving tk shortest path problems. Moreover, together with the boolean facets, these inequalities completely describe $Q^*(C_{tk}^k)$ for $t = 2, 3$ and every $k \geq 2$. However, this is not the case for $t \geq 4$.

The new results in this work allow us to provide a characterization of non boolean facet defining inequalities for $Q^*(C_{4k}^k)$ different from 5-inequalities ($(t+1)$-inequalities with $t = 4$) and to prove that they can be separated in polynomial time.

Recall that, from Theorem 3 and Remark 1, every non boolean facet defining inequality of $Q^*(C_{tk}^k)$ is associated with a simple circuit Γ in $D(C_{tk}^k)$ without $(+1)$-arcs, and has the form

$$(r+1)\sum_{i \in W} x_i + r \sum_{i \notin W} x_i \geq r \left\lceil \frac{s}{p} \right\rceil, \tag{6}$$

where the cardinality s of the set of row arcs in $E(\Gamma)$ is at least 5, and the winding number p of Γ is at least 2. Moreover, Theorem 1 states that the roots of (6) have cardinality $\left\lceil \frac{s}{p} \right\rceil$ or $\left\lceil \frac{s}{p} \right\rceil - 1$.

Let us consider inequalities (6) with $\left\lceil \frac{s}{p} \right\rceil \geq t+2$. Otherwise, the inequality is either not valid, or dominated by the rank inequality, or a $(t+1)$-inequality.

It is easy to see that, for every $i \in [tk]$, $x^i = \{i + jk, j \in \{0, 1, \dots, t-1\}\}$ is a minimum cover of C^k_{tk} and $x^i = x^{i+jk}$ for $j \in [t-1]$. Observe that if we display the elements of $[tk]$ in t rows of k elements , then each column i corresponds to the minimum cover x^i (see Figure 1, for the case $t = 4$).

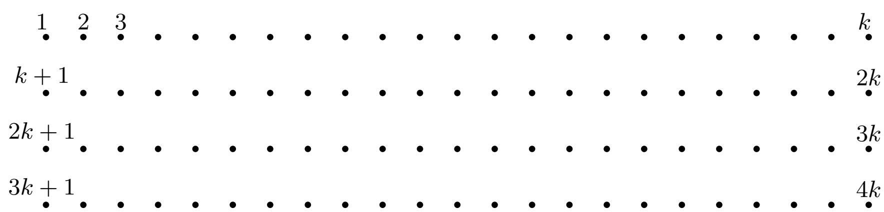

Fig. 1. $[4k]$ elements displayed in 4 rows of k elements

We have the following result:

Theorem 8. *Let $t \geq 4$ and $k \geq 2$. Consider a facet defining inequality for $Q^*(C^k_{tk})$ of the form (6), with $\left\lceil \frac{s}{p} \right\rceil \geq t + 2$. Then, $2r + 1 \leq \left| x^i \cap W \right| \leq t - 1$, for all $i \in [k]$, $r \leq \left\lfloor \frac{t-2}{2} \right\rfloor$ and $\left\lceil \frac{s}{p} \right\rceil \leq 2(t-1)$.*

Proof. Recall that (6) is associated with a circuit in $D(C^k_{tk})$ without (+1)-cycles.

It is not hard to see that, if there exists $x^i \subset W$, then $W = [tk]$ and the inequality is dominated by the rank constraint. Hence, assume $\left| x^i \cap W \right| \leq t - 1$, for all $i \in [k]$.

Let $i \in [k]$. Since x^i has to satisfy (6) and x^i is not a root, we have that

$$t - 1 \geq \left| x^i \cap W \right| > r \left(\left\lceil \frac{s}{p} \right\rceil - |x^i| \right) \geq r(t + 2 - t) = 2r,$$

or equivalently,

$$t - 1 \geq \left| x^i \cap W \right| \geq r \left(\left\lceil \frac{s}{p} \right\rceil - |x^i| \right) + 1 \geq 2r + 1.$$

Then, $r \leq \left\lfloor \frac{t-2}{2} \right\rfloor$ and $\left\lceil \frac{s}{p} \right\rceil \leq 2(t-1)$. □

In particular, for the cases $t = 4, 5$, we have the following:

Corollary 1. *Let $t \in \{4, 5\}$. Then, every non boolean non rank facet defining inequality of $Q^*(C^k_{tk})$ is of the form*

$$2 \sum_{i \in W} x_i + \sum_{i \notin W} x_i \geq \left\lceil \frac{s}{p} \right\rceil . \tag{7}$$

Moreover, assume that (7) is not a $(t+1)$-inequality. Then, if $t = 4$, $\left\lceil \frac{s}{p} \right\rceil = 6$, and if $t = 5$, $\left\lceil \frac{s}{p} \right\rceil \in \{7, 8\}$.

In the following, we strengthen the previous results by characterizing the subsets W in (7) for the case $t = 4$.

Given $i_0 \in [4k]$, we denote by *an* 11-*sequence based on* i_0 any set of the form $\{i_0 = i_{11}, i_1, \ldots, i_{10}\} \subset \{i_0, i_0 + 1, \ldots, i_0 + k - 1\}$ with $i_j < i_{j+1}$ for all $j = 0, \ldots, 9$.

We have the following result:

Theorem 9. *Consider a facet defining inequality for* $Q^*(C^k_{4k})$ *of the form* (6)*, with* $\left\lceil \frac{s}{p} \right\rceil \geq 6$*. Then, the inequality is of the form*

$$2 \sum_{i \in W} x_i + \sum_{i \notin W} x_i \geq 6. \tag{8}$$

Moreover, $\left|x^i \cap W\right| = 3$ *for all* $i \in [k]$ *and there exist* $i_0 \in [4k]$ *and an 11-sequence based on* i_0 *such that*

$$[4k] \setminus W = \bigcup_{t=0}^{10} [i_t + tk, (i_{t+1} - 1) + tk].$$

Figure 2 depicts a possible subset W associated with a facet defining inequality of $Q^*(C^k_{4k})$ having the form (8). Observe that elements not in W induce *11 steps* in the picture. For this reason, we call inequalities of the form (8) as *11-step inequalities*. Moreover, this particular structure allows us to separate 11-step inequalities in polynomial time.

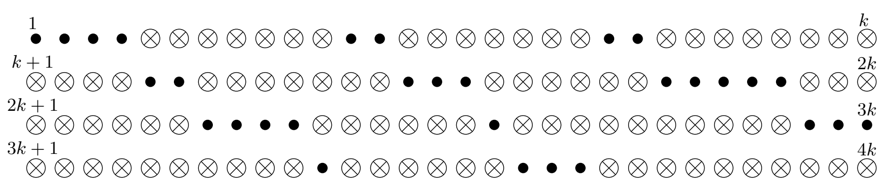

Fig. 2. 11 steps structure. The elements of [4k] belonging to W are represented by crosses, the remaining elements are shown as bullets.

5.1 Separation routines and compact extended formulations

For $x \in \mathbb{R}^{4k}$, define $L(x) := 6 - \sum_{i \in [4k]} x_i$. Given $\hat{x} \in \mathbb{R}^{4k}$, the separation problem for the class of inequalities (8) can be stated as deciding if there exist $i_0 \in [4k]$ and an 11-sequence based on i_0 such that $\sum_{i \in W} \hat{x}_i < L(\hat{x})$ holds for the corresponding set $W \subset [4k]$.

We have the following result:

Theorem 10. *Given $\hat{x} \in \mathbb{R}^{4k}$ and $i_0 \in [4k]$, the problem of deciding if there exists an 11-sequence based on i_0 such that $\sum_{i\in W} \hat{x}_i < L(\hat{x})$ holds for the corresponding set W can be polynomially reduced to a shortest path problem.*

Proof. Define the network $D(i_0) := (V(i_0), A(i_0))$ where $V(i_0) := \{i_0, t\} \cup \left(\bigcup_{i=1}^{10} V^i(i_0)\right)$, and $V^i(i_0) = \{j^{(i)} : j \in [i_0+i, i_0+i+k-11]\}$, for $i = 1, \ldots, 10$. The set $A(i_0)$ is defined by specifying the the sets $N_+(v)$ of successor nodes, for all $v \in V(i_0)$, as follows:

- $N^+(i_0) = V^1(i_0)$,
- For $i = 1, \ldots, 9$ and $j^{(i)} \in V^i(i_0)$, $N^+(j^{(i)}) = \{p^{(i+1)} : p \in [j+1, i_0 + (i+1) + k - 11]\}$,
- For $j^{(10)} \in V^{10}$, $N^+(j^{(10)}) = \{t\}$,
- $N^+(t) = \emptyset$.

Figure 3 illustrates this construction for $i_0 = 1$.

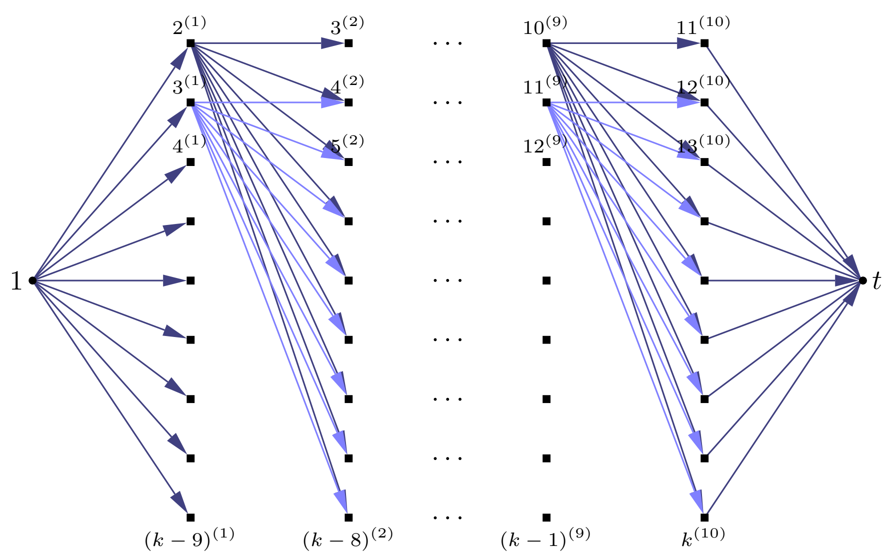

Fig. 3. $D(1) = (V(1), A(1))$.

Finally, arc costs are defined by:

- $c(i_0, j^{(1)}) = \sum_{p=i_0}^{j-1} (\hat{x}_{p+k} + \hat{x}_{p+2k} + \hat{x}_{p+3k})$,
- For $i = 1, \ldots, 9$, $c(j^{(i)}, \ell^{(i+1)}) = \sum_{p=j}^{\ell-1} (\hat{x}_{p+(i-1)k} + \hat{x}_{p+(i+1)k} + \hat{x}_{p+(i+2)k})$,
- $c(j^{(10)}, t) = \sum_{p=j}^{i_0+k-1} (\hat{x}_p + \hat{x}_{p+k} + \hat{x}_{p+3k})$.

It is no hard to see that there is a one-to-one correspondence between (i_0, t)-paths in $D(i_0)$ and 11-sequences of $[4k]$ based on i_0. Moreover, the length of an

(i_0, t)-path in $D(i_0)$ is exactly $\sum_{i \in W} \hat{x}_i$, for the $W \subset [4k]$ associated with the corresponding 11-sequence based on i_0. $\square$

As a consequence of Theorem 10, the separation problem of 11-step inequalities of $Q^*(C_{4k}^k)$ can be reduced to at most $4k$ shortest path problems.

In recent years, an active topic of research in the area of polyhedral combinatorics has been to find *compact extended formulations* of polynomial problems, i.e., formulations of the problem such that, with the addition of a polynomial number of *extra variables*, require only a polynomial number of inequalities. A compact extended formulation for the SCP of matrices C_{tk}^k with $s = 2, 3, 4$ and $k \geq 2$ can be obtained by using the following general property.

Let $Q = \{x \in \mathbb{R}^n : A^1 x \leq b^1, A^2 x \leq b^2\}$, where A^1 has a polynomial number of rows and there exist m polynomially sized linear programs

$$(\mathrm{LP}^i) \;\; z^i_{\mathrm{opt}}(x) = \min\{c^i(x) z^i : M^i z^i \geq d^i \;,\; z^i \in \mathbb{R}^{p^i}\},$$

with $c^i(x)$ a linear function, for $i = 1, \ldots, m$, and linear functions $L^i(x)$, $i = 1, \ldots, m$, such that, given $\hat{x} \in \mathbb{R}^n$,

$$A^2 \hat{x} \leq b^2 \text{ if and only if } z^i_{opt}(\hat{x}) \geq L^i(\hat{x}), \; \forall i = 1, \ldots, m.$$

Then, Q is the projection on x's variables of

$$\begin{aligned} Q^{\mathrm{ext}} = \{(x, y^1, \ldots, y^m) \in \mathbb{R}^n \times \mathbb{R}^{r^1} \times \cdots \times \mathbb{R}^{r^m} : \\ & A^1 x \leq b^1, \\ & (M^i)^T y^i \leq c^i(x), \forall\, i = 1, \ldots, m, \\ & d^i y^i \geq L^i(x), \forall\, i = 1, \ldots, m, \\ & y^i \geq 0, \forall\, i = 1, \ldots, m\}. \end{aligned}$$

In fact, for a given x, by duality,

$$z^i_{\mathrm{opt}}(x) = \max\{d^i y^i : (M^i)^T y^i \leq c^i(x); y^i \geq 0\}.$$

Then, $z^i_{\mathrm{opt}}(x) \geq L^i(x)$ if and only if there exists $y^i \geq 0$ such that $(M^i)^T y^i \leq c^i(x)$ and $d^i y^i \geq L^i(x)$. Clearly, if m is polynomial on n, Q^{ext} is a compact extended formulation of Q.

Let us consider that Q is the set covering polyhedron of C_{tk}^k with $t = 2, 3, 4$ and $k \geq 2$. In this case, the system $A^1 x \leq b^1$ corresponds to the boolean constraints.

If $t = 2, 3$, the system $A^2 x \leq b^2$ corresponds to the $(t+1)$-inequalities which can be separated in polynomial time by solving tk shortest path problems [5].

If $t = 4$, the system $A^2 x \leq b^2$ corresponds to the 5-inequalities and the 11-steps inequalities. In this case, we can separate these inequalities in polynomial time by solving at most $8k$ shortest path problems ([5] and Theorem 9).

According to the above discussion, the linear program formulation of the mentioned shortest path problems allow us to obtain Q^{ext}, a compact extended formulation for the SCP of matrices C_{tk}^k with $t = 2, 3, 4$ and $k \geq 2$.

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6 Appendix: Selected omitted proofs

Proof (of Theorem 4).

Let $A(\Gamma)$ be the submatrix of A whose rows correspond to the row arcs in Γ, $S(\Gamma)$ be the essential nodes of Γ and $NS(\Gamma) = [n] \setminus S(\Gamma)$. We will prove that matrix $A(\Gamma)/NS(\Gamma)$ has $|F|$ rows.

It is clear that if $i \in NS(\Gamma)$ then either $i \in \otimes(\Gamma)$ and $\{i+1, i, i-1\}$ is a path of Γ, or $i \in \bullet(\Gamma)$ but not in Γ, or $i \in \circ(\Gamma)$.

Consider the case $\{i, i+1, \dots, i+h\} \subset \circ(\Gamma)$ and $i-1 \notin \circ(\Gamma)$ and $i+h+1 \notin \circ(\Gamma)$. Then it is not hard to see that $i-1 \in \bullet(\Gamma)$ and $i+h+1 \in \bullet(\Gamma)$. So $(i-1, i) \in E(\Gamma)$ and $(i-1-k_t, i-1) \in E(\Gamma)$ for some row C^t of A with $t \in F$ such that $i = l_t + k_t$. Also, $(i+h, i+h+k_s)$ belongs to $E(\Gamma)$ for row C^s with $s = i+h+1$. In other words, we have the path $(i-1-k_t, i-1, i, i+1, \dots, i+h, i+h+k_s)$ in Γ. From Theorem 2, every node from i to $i+h$ belongs to exactly $p-1$ rows of the set $\{C^j : j \in F\}$. As we can see none of the nodes in the sequence can be the first or the last in any of the rows in $\{C^j : j \in F\}$. Hence if a row in $\{C^j : j \in F\}$ contains a node in the sequence $\{i, i+1, \dots, i+h\}$ then it contains them all. So, if we delete the sequence in the $p-1$ rows, then there is no domination.

Now suppose that $\{i, i-1, \dots, i-h\} \subset \otimes(\Gamma)$ and $i+1 \notin \otimes(\Gamma)$ and $i-h-1 \notin \otimes(\Gamma)$. Then we have that $i+1 \in \bullet(\Gamma)$ and $i-h-1 \in \bullet(\Gamma)$. Then there must be $(i-h-1, i-h-1+k_j) \in E(\Gamma)$ corresponding to the row $C^j = \{i-h, \dots, i-h+k_j-1\}$ where $l_j = i-h$ and another $(i-k_v, i) \in E(\Gamma)$ corresponding to the row $C^v = \{i-k_v+1, \dots, i\}$ where $l_v = i-k_v+1$. From theorem 2, every node from $i-h$ to i belongs to exactly $p+1$ rows indexed in F. Again we can check that if a row indexed in F contains a node in the sequence $\{i-h, \dots, i-1\}$ then it contains them all. So, if we delete the sequence in the $p+1$ rows, then there is no domination. Finally if $i \in \bullet(\Gamma)$ but it is not in Γ it can be deleted in all the p rows where it belongs and there is no domination after deletion.

Let define $A'(\Gamma)$ the matrix obtained after removing all the nonessential nodes in Γ. Now we define the digraph $D(A'(\Gamma))$ whose nodes correspond to the essential nodes in Γ and the arcs correspond to (-1)-arcs and the row arcs in Γ except of the case when there is a sequence of circles, say $\{i, i+1, \dots, i+h\} \subset \circ(\Gamma)$. In this case we know that $i-1 \in \bullet(\Gamma)$ and $i+h+1 \in \bullet(\Gamma)$, then we replace the path $(i-1, i, i+1, \dots, i+h, i+h+k_s)$ in Γ by the row arc $(i-1, i+h+k_s)$.

Let consider the circuit Γ' of $D(A'(\Gamma))$ obtained from Γ after removing all its nonessential nodes in such a way that every cross or bullet in Γ' is a cross or bullet in Γ.

For every $v \in S(\Gamma) \cap \bullet(\Gamma)$, there is a row arc in Γ' leaving v. Therefore $s = |F'| = \bullet(\Gamma)$. It follows that every row in $A'(\Gamma)$ corresponds to a row arc in Γ'. Let $v \in S(\Gamma) \cap \otimes(\Gamma)$. Then the only arcs in $E(\Gamma')$ to whom v belongs are $(v, v-1)$ and $(v-k_i, v)$ with $v = l_i + k_i - 1$ for some $i \in \{1, \dots, s\}$. Observe that $v-1 \in \bullet(\Gamma)$ and it is the tail of the arc $(v-1, v-1+k_j)$ for some $j \in \{1, \dots, s\}$. In addition, the arc $(v-k_i, v)$ in $E(\Gamma')$ corresponds to the row $\{v-k_i+1, \dots, v\} \cap S(\Gamma)$ and the arc $(v-1, v-1+k_j) \in E(\Gamma')$ to the row $\{v, \dots, v+k_j-1\} \cap S(\Gamma)$. If v belongs to other row of $A'(\Gamma)$ it is a middle node of it. If we contract $A'(\Gamma)/v$ there is no dominating row, since there is neither a row arc of Γ' ending at $v-1$ nor row arc of Γ' beginning at v. Hence $A'(\Gamma)/\otimes(\Gamma)$ is a square matrix of order s. Let Γ'' be a circuit obtained from Γ' by replacing the arc $(v-k_i, v)$ by $(v-k_i, v-1)$ every time $v \in \otimes(\Gamma) \cap S(\Gamma)$. It is easy to check that the nodes of Γ'' are bullet nodes and still belong to p rows of $A'(\Gamma)$. Moreover, Γ'' is a circuit that reaches every node in $S(\Gamma) \cap \bullet(\Gamma)$. So $A'(\Gamma)/\otimes(\Gamma)$ is isomorphic to a circulant matrix C_s^p. Moreover, since the circuit is simple we have that $gcd(s,p) = 1$, otherwise there would be a subcircuit in Γ''. □

Proof (of Theorem 5). Let $\mathrm{rfi}(\Gamma)$ be the facet of $Q_I(C_n^k)$ corresponding to Γ. Suppose that $\circ(\Gamma) \neq \emptyset$. From definition of the $\mathrm{rfi}(\Gamma)$ it follows that $\otimes(\Gamma) \neq \emptyset$. Also, a sequence of circles in Γ begins with a bullet, and a sequence of crosses in Γ ends in a bullet.

We say that a path P in Γ has the $(\otimes \bullet \circ)$-property if it is of the form $(v, v-1, v-1+k, v-1+2k, \ldots, v-1+hk, v+hk)$ for some $h \geq 1$. It is clear that $v \in \otimes(\Gamma)$, $v+hk \in \circ(\Gamma)$ and $v-1+lk \in \bullet(\Gamma)$ for $l = 0, \ldots, h$.

It is straightforward to check there is such a path in Γ. Assume that P has the $(\otimes \bullet \circ)$-property and the smallest possible value h.

Let Φ be the set of arcs obtained from Γ after replacing P by the path $(v, v+k, \ldots, v+hk)$.

Claim *Φ is a simple circuit.*

Proof (of the claim). In order to prove that Φ is a simple circuit, we need to show that $v+lk \notin V(\Gamma)$ for every $l = 1, \ldots, h-1$.

Observe that if $v+jk \in V(\Gamma)$ for some $1 \leq j \leq h-1$ then $v+jk \notin \otimes(\Gamma)$ since $(v+jk, v-1+jk) \notin \Gamma$. Also, $v+k \notin \circ(\Gamma)$ since $(v-1+jk, v+jk) \notin E(\Gamma)$. Hence $v+jk \in \bullet(\Gamma)$.

Then, let j be the smallest number such that $1 \leq j \leq h$ and $v+jk \in V(\Gamma)$. If $j = h$ then Φ is a simple circuit and the claim follows.

Assume that $1 \leq j \leq h-1$. Then the only possible arc of $E(\Gamma)$ that reaches the node $v+jk$ is the arc $(v+1+jk, v+jk)$ and then $(v+jk, v+(j+1)k) \in E(\Gamma)$. If the circuit Γ continues from $v+(j+1)k$ with arcs of length k it arrives at node $v+hk$. But this node has already been achieved by Γ. So, there must be $j+1 \leq m \leq h-1$ such that $(v+mk, v+1+mk) \in E(\Gamma)$. Hence $v+1+mk \in \circ(\Gamma)$ and there is a path $P' = (v+1+jk, v+jk, v+jk, v+(j+1)k, \ldots, v+mk, v+1+mk)$ in Γ. But, P' has the $(\otimes\bullet\circ)$-property with $m-j < h$. Then P is not the shortest path with the property and the claim follows. □

So, Φ is a simple circuit with the same number of row arcs as Γ. Also, $|\circ(\Phi)| = |\circ(\Gamma)| - 1$ and $|\otimes(\Phi)| = |\otimes(\Gamma)| - 1$. Hence the $\mathrm{rfi}(\Gamma)$ is dominated by $\mathrm{rfi}(\Phi)$ and the theorem follows. □

Proof (of Theorem 6). Let us call N the set of all non essential nodes in (Γ) plus the essential crosses in (Γ), i.e. $N = NS(\Gamma) \cup \otimes(\Gamma)$.

According to Theorem 4, if we delete all nodes in N in every row C^i with $i \in F$, it holds that $\left|C^i \cap S(\Gamma) \cap \bullet(\Gamma)\right| = p$. So, if we show that $\left|C^i \cap S(\Gamma) \cap \bullet(\Gamma)\right| \geq$

p for every $i \notin F$, we get that C_n^k/N is isomorphic to C_s^p. And also we obtain that $gcd(s,p)=1$.

Suppose there is a row C^{z+1} with $z+1 \notin F$, such that

$$|\{z+1,\ldots,z+k\} \cap S(\Gamma) \cap \bullet(\Gamma)| \leq p-1.$$

Then $(z,z+k) \notin E(\Gamma)$. From Lemma 5 $\circ(\Gamma)=\emptyset$ and then $z \notin \bullet(\Gamma) \cap S(\Gamma)$.

Let us call $\{u_1,u_2,\ldots,u_l\} = \{z+1,\ldots,z+k\} \cap S(\Gamma) \cap \bullet(\Gamma)$ with $1 \leq l \leq p-1$.

Let u_0 be the first essential bullet that precedes z and then $(u_0,u_0+k) \in E(\Gamma)$. But u_0+k precedes $z+k$ and $u_0 \neq z$ then

$$\{u_0+1,\ldots,u_0+k\} \cap S(\Gamma) \cap \bullet(\Gamma) \subseteq \{u_1,u_2,\ldots,u_l\}.$$

Hence $\left|C^{u_0+1} \cap S(\Gamma) \cap \bullet(\Gamma)\right| \leq l \leq p-1$ contradicting Theorem 4. Thus $N = NS(\Gamma) \cup \otimes(\Gamma)$ is such that C_n^k/N is isomorphic to a circulant matrix C_s^p with $gcd(s,p)=1$.

In [4], the minor inequality associated with C_s^p is obtained from the rows C^{i+1} for every $i \notin N$. But, $i \notin N$ if and only if $i \in S(\Gamma) \cap \bullet(\Gamma)$. Hence $i \notin N$ if and only if $(i,i+k) \in E(\Gamma)$. The family of rows involved in obtaining the minor inequality coincides with the rows indexed in F. Also in [4] it was proved that every $i \in W$ belongs to exactly $p+1$ rows in F. Hence $W = \otimes(\Gamma)$. As a consequence every rfi defining a facet of $Q^*(C_n^k)$ is a minor inequality. $\square$

Proof (Theorem 9). From Corollary 1 we know that every non boolean defining a facet of $Q^*(C_{4k}^k)$ different from a 5-inequality, is of the form

$$2\sum_{i\in W} x_i + \sum_{i\notin W} x_i \geq 6. \tag{9}$$

From Theorem 8 we know that, for every minimum covers x^i with $i \in [k]$, $|x^i \cap W| = 3$ or, equivalently, $|x^i \setminus W| = 1$.

First of all, we need to prove that if $x^i \setminus W = \{i+jk\}$ for some $j \in \{0,1,2,3\}$, then $x^{i+1} \setminus W = \{i+1+jk\}$ or $x^{i+1} \setminus W = \{i+1+(j+1)k\}$.

Let Γ be a circuit associated with (9). Then, $\otimes(\Gamma) = W$. W.l.o.g. we can assume that $x^i \setminus \otimes(\Gamma) = \{i+3k\}$. We will prove that $x^{i+1} \setminus \otimes(\Gamma) = \{i+1+3k\}$ or $x^{i+1} \setminus \otimes(\Gamma) = \{i+1\}$.

In fact, the only way to reach $i+k$ is through the arc $(i+1+k, i+k)$. In fact, if $(i,i+k) \in E(\Gamma)$, Γ would not be a circuit and if $(i-1+k,i+k) \in E(\Gamma)$, Γ would not be simple. Therefore, $i+1+k \in \otimes(\Gamma)$. With the same reasoning, $i+1+2k$ and $i+2+2k$ belong to $\otimes(\Gamma)$.

Observe that the row arcs in Γ correspond to the indices in $F = \{i \in [4k] : i \notin W, i-1 \in W\}$. Then, inequality (9) is a rfi corresponding to F and $p = \max_{j\in[4k]} \sum_{i\in F} a_{ij} - 1 \geq 2$. If $s=|F|$, we have that $\left\lceil \frac{s}{p} \right\rceil = 6$. Since s is not a multiple of p, we have that $s = 5p+1$.

Let A^j be j-th column of C_{4k}^k. Observe that $A^i = C^{j-k+1}$ and, for all minimum cover $\{i, i+, k, i+2k, i+3k\}$, the columns $A^i, A^{i+k}, A^{i+2k}, A^{i+3k}$ define a partition of $[4k]$.

Let $i \in [4k]$ and w.l.o.g. suppose that $x^i \cap W = \{i, i+, k, i+2k\}$. Since $W = \{j \in [4k] : \sum_{i \in F} a_{ij} = p+1\}$,

$$\sum_{t \in F} a_{tj} = p+1 \text{ for } j \in \{i, i+k, i+2k\} \tag{10}$$

$$\text{and} \quad \sum_{t \in F} a_{t(i+3k)} \leq p. \tag{11}$$

Since $|F| = 5p+1$ and by (10), $\left|F \cap A^i\right| = \left|F \cap A^{i+k}\right| = \left|F \cap A^{i+2k}\right| = p+1$, we have that $\left|F \cap A^{i+3k}\right| = 5p+1-(3p+3) = 2p-2$. By (11), $\left|F \cap A^{i+3k}\right| \leq p$, implying $p \leq 2$. Therefore, we have that $p = 2$ and $s = 11$.

Given $i_0 \in F$, for each $j \in F$ let $i(j) \in [i_0.i_0+k-1]$ such that $j = i(j)$ mod k. Clearly, $\{i(j) : j \in F\}$ is an 11-sequence based on i_0.

□